**Teacher Notes**

# Limits Student Study Session

**Solving limits:** The vast majority of limits questions can be solved by using one of four techniques: SUBSTITUTING, FACTORING, CONJUGATING, or by INSPECTING A SERIES OF TERMS.

AP Calculus questions can be one-sided or two-sided limits, can be given analytically, graphically, or from a table, and can to a value or to positive or negative infinity.

Limits describe what happens as X approaches a certain number (or $\pm\infty$), but do not necessarily tell you what happens when X equals the number.

You should always first try putting the value in for the X and see if the limit is defined. It usually is undefined by this method, so you must first try factoring the numerator and denominator and cancelling out like factors and then substitute in the value. If that fails, and you have radicals in the denominator or numerator, multiply by the conjugate of that Zwork), it works best to substitute 4-5 values and see what value the limit appears to be heading. This method, as crude as it seems, works in some of the more difficult problems. Do not stop after 2-3 value substitutions since you can be deceived on what the limit appears to be if you stop too early.

***When does a limit exist?***

In order for a limit to exist for a function f(x) at some value c, three things must happen:

1. The left hand limit must exist for x=c
2. The right hand limit must exist for x=c
3. The left hand limit and right hand limits at c must be equal.

In other words:

$$\lim_{x\to a^-} f(x) = c = \lim_{x\to a+} f(x) \text{ then } \lim_{x\to a} f(x) = c$$

Limits you should memorize:

$$\lim_{x\to 0} (\sin(x))/x = 1 \qquad \lim_{x\to 0} (1-\cos(x))/x = 0$$

**Student Notes**

***Limits at as x approaches +/-∞***

Typically there is one multiple choice question with a limit as x approaches infinity. These are some of the highest scoring MC questions.

1. If the x with the largest exponent is in the numerator, the value of the limit is +/-∞
2. If the x with the largest exponent is in the denominator, the value of the limit is 0
3. If the x with the largest exponent is the same in the numerator and denominator, the limit is then the coefficients of the two x's with the largest exponent in the numerator and denominator

# *Continuity*

A function is said to be continuous if it has no "breaks" in it. In other words, a function is continuous over an interval [a, b], if , as we trace over the graph, we do not need to take our pencil off the page. A function *f* is continuous at a point (a, *f*(a)) if as x gets close to a, then *f*(a) gets close to *f*(a). Continuity has three conditions:

1. The limit from the left equals the limit from the right
2. The value exists at the point
3. The limit equals the value of the point

## *3 types of discontinuities exist*

1. Point discontinuity
2. Jump discontinuity
3. Infinite discontinuity

**IMPORTANT: Differentiability implies continuity, but continuity does NOT imply differentiability.**

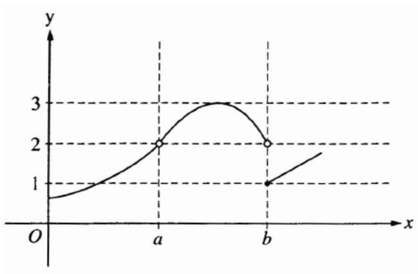
Differentiability means the derivative exists everywhere on its domain. In order to be differentiable, a function must be continuous. So ALL differentiable functions are continuous, but NOT all continuous functions are differentiable. Continuous but not differentiable functions could have cusps (corner points) like the absolute value function f(x) = lxl

# Limits, Continuity and Definition of the Derivative

**Multiple Choice**

*Identify the choice that best completes the statement or answers the question.*

1.




The graph of the function *f* is shown in the figure above. Which of the following statements about *f* is true?

a. $\lim_{x \to a} f(x) = \lim_{x \to b} f(x)$

b. $\lim_{x \to a} f(x) = 2$

c. $\lim_{x \to b} f(x) = 2$

d. $\lim_{x \to b} f(x) = 1$

e. $\lim_{x \to a} f(x)$ does not exist

2. $\lim_{x \to 1} \frac{x}{\ln x}$ is

a. 0

b. $\frac{1}{e}$

c. 1

d. $e$

e. nonexistent

3. $\lim_{x \to \infty} \frac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} =$

a. 4

b. 1

c. $\frac{1}{4}$

d. 0

e. -1

4. If $a \neq 0$, then $\lim_{x \to a} \frac{x^2 - a^2}{x^4 - a^4}$ is

a. $\frac{1}{a^2}$

b. $\frac{1}{2a^2}$

c. $\frac{1}{6a^2}$

d. 0

e. nonexistent

5. If $f(x) = \begin{cases} \ln x & \text{for } 0 < x \leq 2 \\ x^2 \ln 2 & \text{for } 2 < x \leq 4, \end{cases}$ the $\lim_{x \to 2} f(x)$ is

a. $\ln 2$

b. $\ln 8$

c. $\ln 16$

d. 4

e. nonexistent

6. If $f'(x) = \cos x$ and $g'(x) = 1$ for all *x*, and if *f*(0) =*g*(0) = 0 , then $\lim_{x \to 0} \frac{f(x)}{g(x)}$ is

a. $\frac{\pi}{2}$

b. 1

c. 0

d. -1

e. nonexistent

7. If $\lim_{x \to a} f(x) = L$, where *L* is a real number, which of the following must be true?

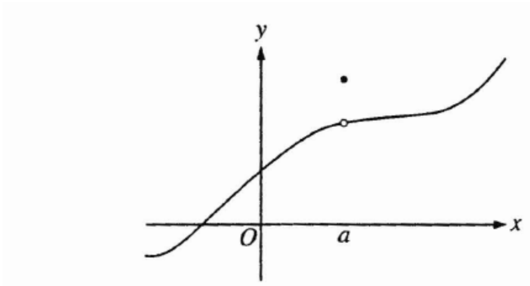
a. $f'(a)$ exists.

b. *f* (*x*) is continuous at *x* = *a*.

c. *f* (*x*) is defined at *x* = *a*.

d. *f* (*a*) =*L*

e. None of the above




8. The graph of a function *f* is shown above. Which of the following statements about *f* is false?

a. *f* is continuous at $x = a$.
b. *f* has a relative maximum at $x = a$.
c. $x = a$ is in the domain of *f*.
d. $\lim_{x \to a^+} f(x)$ is equal to $\lim_{x \to a^-} f(x)$
e. $\lim_{x \to a} f(x)$ exists.

9.

$$f(x) = \begin{cases} x+2 & \text{if } x \le 3 \\ 4x-7 & \text{if } x > 3 \end{cases}$$

Let *f* be the function given above. Which of the following statements are true about *f* ?

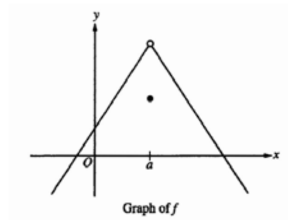
I. $\lim_{x \to 3} f(x)$ exists
II. *f* is continuous at $x = 3$.
III. *f* is differentiable at $x = 3$.

a. None
b. I only
c. II only
d. I and II only
e. I, II, and III

10. $\lim_{x \to 0} (x \csc x)$ is

a. $-\infty$
b. -1
c. 0
d. 1
e. $\infty$

11.




The graph of the function *f* is shown above. Which of the following statements must be false?

a. *f(a)* exists.
b. *f(x)* is defined for $0 < x < a$.
c. *f* is not continuous at $x = a$.
d. $\lim_{x \to a} f(x)$ exists.
e. $\lim_{x \to a} f'(x)$ exists

12. Let *f* be a function such that $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 5$.
Which of the following must be true?
I. *f* is continuous at x = 2.
II. *f* is differentiable at x = 2.
III. The derivative of *f* is continuous at x = 2.

a. I only
b. II only
c. I and II only
d. I and III only
e. II and III only

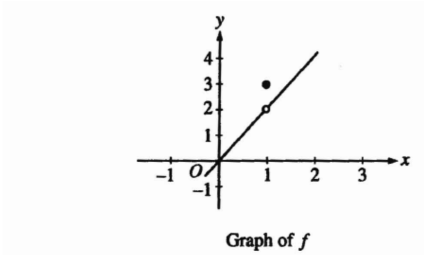
13. $\lim_{n \to \infty} \frac{3n^3 - 5n}{n^3 - 2n^2 + 1}$ is

a. -5
b. -2
c. 1
d. 3
e. nonexistent

14. If the function *f* is continuous for all real numbers and if $f(x) = \frac{x^2 - 4}{x + 2}$ when $x \ne 2$, then *f*(-2)=

a. -4
b. -2
c. -1
d. 0
e. 2

15.




The graph of the function $f$ is shown in the figure above. The value of $\lim_{x \to 1} \sin(f(x))$ is

a. 0.909
b. 0.841
c. 0.141
d. -0.416
e. nonexistent

16. $\lim_{n \to \infty} \frac{4n^2}{n^2 + 10{,}000n}$ is

a. 0
b. $\frac{1}{2{,}500}$
c. 1
d. 4
e. nonexistent

17. If $\begin{cases} f(x) = \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}, \text{ for } x \neq 2, \\ f(2) = k \end{cases}$ and if $f$ is continuous at $x = 2$ , then $k =$

a. 0
b. $\frac{1}{6}$
c. $\frac{1}{3}$
d. 1
e. $\frac{7}{5}$

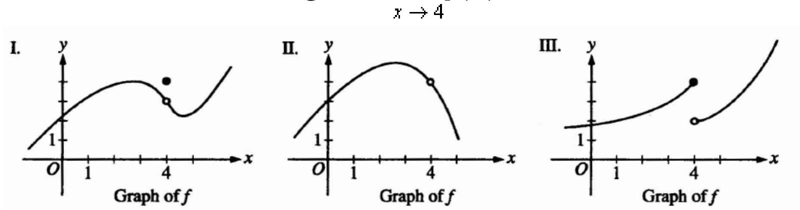
18. $\lim_{\theta \to 0} \frac{1 - \cos\theta}{2\sin^2\theta}$ is

a. 0
b. $\frac{1}{8}$
c. $\frac{1}{4}$
d. 1
e. nonexistent

19. For $x \geq 0$, the horizontal line $y = 2$ is an asymptote for the graph of the function $f$ Which of the following statements must be true?

a. $f(0) = 2$
b. $f(x) \neq 2$ for all $x \geq 0$
c. $f(2)$ is undefined
d. $\lim_{x \to 2} f(x) = \infty$
e. $\lim_{x \to \infty} f(x) = 2$

20. For which of the following does $\lim_{x \to 4} f(x)$ exist?




a. I only
b. II only
c. III only
d. I and II only
e. I and III only

# Limits, Continuity and Definition of the Derivative
# Answer Section

## MULTIPLE CHOICE

1. ANS: B DIF: Medium MSC: 59% answered correctly
   NOT: 1997 #15
2. ANS: E DIF: Hard MSC: 34% answered correctly
   NOT: 1997 #21
3. ANS: C
   Since the polynomials in the numerator and denominator are of the same degree, the limit as *x* approaches infinity is the ratio of their leading coefficients $\frac{1}{4}$.

   DIF: Easy MSC: 82% answered correctly NOT: 2003 #6
4. ANS: B
   $$\lim_{x \to a} \frac{x^2 - a^2}{x^4 - a^4} = \lim_{x \to a} \frac{x^2 - a^2}{\left(x^2 - a^2\right)\left(x^2 + a^2\right)} = \lim_{x \to a} \frac{1}{x^2 + a^2} = \frac{1}{2a^2}$$

   Answer B

   DIF: Medium REF: Calculator allowed but not required
   MSC: 45% answered correctly NOT: 1998 #83
5. ANS: E
   $\lim_{x \to 2+} f(x) = 2^2 \ln(2)$ and $\lim_{x \to 2-} f(x) = \ln(2)$. since these are not equal there is no limit as *x* approaches 2.

   Answer E

   DIF: Medium REF: No calculator allowed MSC: 52% answered correctly
   NOT: 1998 #12
6. ANS: B
7. ANS: E
8. ANS: A
   The interesting feature of this function is that one pont has been removed and moved above the graph. So

   A is FALSE
   B is TRUE because the value at $x = a$ is larger than those with *x*-values close to *a*.
   C is TRUE because there is a function value at $x = a$.
   D and E are TRUE because as *x* approaches *a* the values near $x = a$ are all approaching the same value, the *y*-coordinate of the hole.

   Answer A.

   DIF: Medium REF: Calculator allowed but not required
   MSC: 61% answered correctly NOT: 1998 #76
9. ANS: D

At $x = 3$, both parts of the function's rule give the same value (5). Therefore, the function is continuous and the limit in I exists (= 5). However, the slope change abruptly at $x = 3$ indicating the function is not differentialbe there.

DIF: Medium MSC: 45% answered correctly NOT: 2003 #20

10. ANS: D
11. ANS: E DIF: Easy MSC: 75% answered correctly
NOT: 2003 #76
12. ANS: C
The limit is the definition of the derivative of $f$ at $x = 2$.
Since the derivative exists, $f$ is continuous at $x = 2$. I is TRUE
Since the derivative exists at $x = 2$, $f$ is differentiable there. II is TRUE
There is not enought information to say whether this is true. Not every differentiable function has a continuous derivative. III is NOT TRUE. The counterexample is

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

See also 1982 BC FR #7 and 1962 FR #6 and the 2007 "Practice Exam" BC #79
Answer C

DIF: Medium REF: Calculator allowed but not required
MSC: 40% answered correctly NOT: 1997 #79

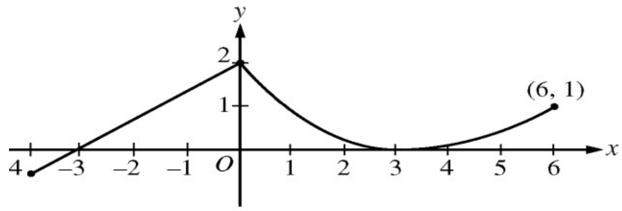
13. ANS: D
14. ANS: A
15. ANS: A DIF: Medium MSC: 62% answered correctly
NOT: 2003 #81
16. ANS: D
17. ANS: B
18. ANS: C
19. ANS: E
Choice E is the analytic expression that corresponds to a horizontal asymptote.
A, B, C and D are false becasue it is possilbe that at or "near" x = 0, the function value is 2. horizontal asymptotes describe end behavior of functions, not behavior "near" the origin (x = 2 is "near" the origin)

DIF: Medium MSC: 41% answered correctly NOT: 2003 #3

20. ANS: D
In III the graph has a jump discontinuity at $x = 4$, ther limit from each side is different, therefore the limit does not exist.
In I and II the functions have a "removable discontiuity" at $x = 4$. The limit exist in both cases although it is not equal to the value there.

DIF: Medium REF: Calculator allowed but not required
MSC: 55% answered correctly NOT: 2003 #79

## 2009 Form B Question 3




Graph of $f$

A continuous function $f$ is defined on the closed interval $-4 \le x \le 6$.

(a) Is $f$ differentiable at $x = 0$ ? Use the definition of the derivative with one-sided limits to justify your answer.

## 2008 Question 6

Let $f$ be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$.

(d) Find $\displaystyle\lim_{x\to 0^+} f(x)$.

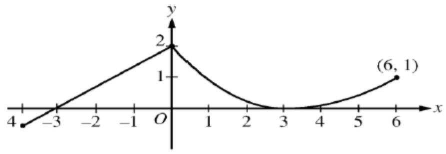
## 2011 Form B Question 2

A 12,000-liter tank of water is filled to capacity. At time $t = 0$, water begins to drain out of the tank at a rate modeled by $r(t)$, measured in liters per hour, where $r$ is given by the piecewise-defined function:

$$r(t) = \begin{cases} \dfrac{600t}{t+3} & \text{for } 0 \le t \le 5 \\ 1000e^{-0.2t} & \text{for } t > 5 \end{cases}$$

(a) Is $r$ continuous at $t = 5$ ? Show the work that leads to your answer.

## 2009 Form B Question 3



Graph of $f$

A continuous function $f$ is defined on the closed interval $-4 \le x \le 6$.

(a) Is $f$ differentiable at $x = 0$ ? Use the definition of the derivative with one-sided limits to justify your answer.

(a) $\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \frac{2}{3}$

$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} < 0$

Since the one-sided limits do not agree, $f$ is not differentiable at $x = 0$.

$2: \begin{cases} 1: \text{sets up difference quotient at } x = 0 \\ 1: \text{answer with justification} \end{cases}$

### 2008 Question 6

Let $f$ be the function given by $f(x) = \frac{\ln x}{x}$ for all $x > 0$.

(d) Find $\lim_{x \to 0^+} f(x)$.

Answer:

(d) $\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$ or Does Not Exist | 1 : answer

## 2011 Form B Question 2

A 12,000-liter tank of water is filled to capacity. At time $t = 0$, water begins to drain out of the tank at a rate modeled by $r(t)$, measured in liters per hour, where $r$ is given by the piecewise-defined function:

$$r(t) = \begin{cases} \frac{600t}{t+3} & \text{for } 0 \le t \le 5 \\ 1000e^{-0.2t} & \text{for } t > 5 \end{cases}$$

(a) Is $r$ continuous at $t = 5$ ? Show the work that leads to your answer.

Answer:

(a) $\lim_{t \to 5^-} r(t) = \lim_{t \to 5^-} \left( \frac{600t}{t+3} \right) = 375 = r(5)$

$\lim_{t \to 5^+} r(t) = \lim_{t \to 5^+} \left( 1000e^{-0.2t} \right) = 367.879$

Because the left-hand and right-hand limits are not equal, $r$ is not continuous at $t = 5$.

**Scoring: 2 Points: Conclusion with analysis**