PTFs #BC 01 – "U-Substitution" Rule

1. Let $u =$ inner function.
2. Find du, then solve for dx.
3. Substitute u & du into the integrand (it should now fit one of the integration rules).
4. Integrate.
5. Substitute the inner function back for u.

---

1. Integrate $\int\left[9\left(x^2+3x+5\right)^8(2x+3)\right]dx$

$u = x^2+3x+5$

$du = (2x+3)\,dx$

$\frac{1}{2x+3}du = dx$

$\int 9u^8 (2x+3)\frac{1}{2x+3}du$

$\int 9u^8 du = \frac{9u^9}{9}+C = \boxed{(x^2+3x+5)^9+C}$

2. Integrate $\int\left(\sin^2 3x\cos 3x\right)dx$

$u = \sin 3x$

$du = 3\cos 3x\,dx$

$\frac{1}{3\cos 3x} = dx$

$\int u^2 \cos 3x \cdot \frac{1}{3\cos 3x}du$

$\int \frac{1}{3}u^2 du = \frac{u^3}{9}+C = \boxed{\frac{\sin^3(3x)}{9}+C}$

3. Integrate $\int e^{3x+1}dx$

$\int e^{3x+1}dx = \boxed{\frac{1}{3}e^{3x+1}+C}$

4. Integrate $\int\frac{e^{\tan x}}{\cos^2 x}dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$\cos^2 x\,du = dx$

$\int \frac{e^u}{\cos^2 x}\cdot\cos^2 x\,du = \int e^u du = e^u + C$

$= \boxed{e^{\tan x}+C}$

5. Integrate $\int\frac{e^x}{1+e^x}dx$

$u = 1+e^x$

$du = e^x\,dx$

$\frac{1}{e^x}du = dx$

$\int \frac{e^x}{u}\frac{1}{e^x}du = \int\frac{1}{u}du = \ln|u|+C$

$= \boxed{\ln|1+e^x|+C}$

6. If $\frac{dy}{dx} = \cos(2x)$, then $y = \underline{\frac{1}{2}\sin(2x)+C}$?

PTFs #BC 02 – Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

1. Choose "u" by using LIPET as a guide.
2. Differentiate u to find du.
3. Integrate dv to find v.
4. May need to repeat the process if the new integral is still not possible.
5. If u is a 2nd degree polynomial or higher, use the tabular method to solve.

---

1. $\int xe^{2x}dx$

$f(x)=x \qquad g'(x)=e^{2x}$

$f'(x)=1 \qquad g(x)=\frac{1}{2}e^{2x}$

$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$

$= \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$

2. $\int x\sec^2 x\,dx$

$f(x)=x \qquad g'(x)=\sec^2 x$

$f'(x)=1 \qquad g(x)=\tan x$

$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$= x\tan x - \ln|\sec x| + C$

$= \boxed{x\tan x + \ln|\cos x| + C}$

3. $\int \ln x\,dx$

$f(x)=\ln x \qquad g'(x)=1$

$f'(x)=\frac{1}{x} \qquad g(x)=x$

$\int \ln x\,dx = x\ln x - \int 1\,dx$

$= \boxed{x\ln x - x + C}$

4. $\int x^2 \sin x\,dx$

f	g'
$+x^2$	$\sin x$
$-2x$	$-\cos x$
$+2$	$-\sin x$
-0	$\cos x$

$\boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}$

5. $\int \tan^{-1}(x)dx$

$f(x)=\tan^{-1}x \qquad g'(x)=1$

$f'(x)=\frac{1}{1+x^2} \qquad g(x)=x$

$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}dx$

$u=1+x^2$

$du=2x\,dx$

$\frac{1}{2x}du = dx$

$\int \frac{x}{u}\cdot\frac{1}{2x}du$

$=\int \frac{1}{2}\frac{1}{u}du$

$=\frac{1}{2}\ln|u|+C$

$=\frac{1}{2}\ln|1+x^2|+C$

$= \boxed{x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C}$

Use partial fractions when the numerator of the rational function is NOT a multiple of the derivative of the denominator.

6. Factor the denominator.
7. Decompose the fraction.
8. Integrate.

*The fraction has to be bottom heavy, if not, use long division to find the remainder. Integrate the quotient as usual and then use partial fractions on the remainder.

---

1. $\int\left(\frac{5x-3}{x^2-2x-3}\right)dx$

$\int \frac{5x-3}{(x-3)(x+1)}dx = \int \frac{A}{x+1} + \frac{B}{x-3}$

$A(x-3) + B(x+1) = 5x-3$

$B(4) = 20-3$ $\quad$ $A(-4) = -5-3$

$B = \frac{17}{4}$ $\quad$ $A = 2$

$\int \frac{5x-3}{(x-3)(x+1)}dx = \int \frac{2}{x+1} + \frac{17/4}{x-3}dx$

$= 2\ln|x+1| + \frac{17}{4}\ln|x-3| + C$

2. $\int_2^3 \left(\frac{3}{(x-1)(x+2)}\right)dx$

$\int \frac{3}{(x-1)(x+2)}dx = \int \frac{A}{x-1} + \frac{B}{x+2}dx$

$A(x+2) + B(x-1) = 3$

$B(-3) = 3$ $\quad$ $A(3) = 3$

$B = -1$ $\quad$ $A = 1$

$\int_2^3 \frac{3}{(x-1)(x+2)}dx = \int_2^3 \frac{1}{x-1} + \frac{-1}{x+2}dx$

$= \ln|x-1| - \ln|x+2| \Big|_2^3$

$= \ln\left|\frac{x-1}{x+2}\right| \Big|_2^3$

$= \ln\frac{2}{5} - \ln\frac{1}{3}$

$= \ln\frac{6}{5}$

- $\int_{\#}^{\infty} f(x)dx = \lim_{a\to\infty}\int_{\#}^{a} f(x)dx$
- $\int_{-\infty}^{\#} f(x)dx = \lim_{a\to-\infty}\int_{a}^{\#} f(x)dx$
- $\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_{a}^{\#} f(x)dx + \lim_{b\to\infty}\int_{\#}^{b} f(x)dx$
- If $f(x)$ has a discontinuity at a, then $\int_{a}^{\#} f(x)dx = \lim_{b\to a^+}\int_{b}^{\#} f(x)dx$
- If $f(x)$ has a discontinuity at a, then $\int_{\#}^{a} f(x)dx = \lim_{b\to a^-}\int_{\#}^{b} f(x)dx$
- If $f(x)$ has a discontinuity at a, which is between the interval, then break up the interval into two intervals and evaluate using limits.

Evaluate each integral.

1. $\int_{1}^{\infty} \frac{1}{x^2}dx$

$\lim_{n\to\infty}\int_1^n \frac{1}{x^2}dx = \lim_{n\to\infty} -\frac{1}{x}\Big|_1^n$

$= \lim_{n\to\infty} -\frac{1}{n} + 1 = 0 + 1 = \boxed{1}$

2. $\int_{2}^{5} \frac{1}{\sqrt{x-2}}dx$

$\lim_{n\to2^+}\int_n^5 \frac{1}{\sqrt{x-2}}dx = 2\sqrt{x-2}\Big|_n^5$

$= \lim_{n\to2^+} 2\sqrt{3} - 2\sqrt{n-2} = 2\sqrt{3} - 0$

$= \boxed{2\sqrt{3}}$

3. $\int_{0}^{\infty} x^2 e^{-x^3}dx$

$\lim_{n\to\infty}\int_0^n x^2 e^{-x^3}dx = \lim_{n\to\infty} -\frac{1}{3}e^{-x^3}\Big|_0^n$

$= \lim_{n\to\infty} -\frac{1}{3}e^{-n^3} + \frac{1}{3}$

$= \lim_{n\to\infty} -\frac{1}{3e^{n^3}} + \frac{1}{3} = 0 + \boxed{\frac{1}{3}}$

4. $\int_{0}^{3} \frac{1}{x-1}dx$

$\lim_{n\to1^-}\int_0^n \frac{1}{x-1}dx + \lim_{b\to1^+}\int_b^3 \frac{1}{x-1}dx$

$\lim_{n\to1^-} \ln|x-1|\Big|_0^n$

$\lim_{n\to1^-} \ln|n-1| - \ln 1$

$= -\infty$

$\int_0^3 \frac{1}{x-1}dx$ does not exist or is indefinite

PTFs #BC 05 – Arc Length

Let the function given by $f(x)$ represent a smooth curve on the interval $[a,b]$.
The arc length of f between a and b is:

$$s=\int_a^b\left(\sqrt{1+[f'(x)]^2}\right)dx$$

---

1. Set up an integral to represent the length of $y=x^3$ on [0, 5]. Use your calculator to find the length of the curve.

$L=\int_0^5\sqrt{1+(3x^2)^2}\,dx$

$=\underline{125.680}$

2. Set up an integral to represent the length of the graph of $y=\frac{1}{2}\left(e^x+e^{-x}\right)$ on [0, 3]. Find the length without a calculator.

$y'=\frac{1}{2}(e^x-e^{-x})$

$(y')^2=\frac{1}{4}(e^x-e^{-x})^2$

$L=\int_0^3\sqrt{1+\frac{1}{4}(e^x-e^{-x})^2}\,dx$

$=\int_0^3\frac{1}{2}\sqrt{4+(e^x-e^{-x})^2}\,dx$

$=\int_0^3\frac{1}{2}\sqrt{4+e^{2x}-2+e^{-2x}}\,dx$

$=\int_0^3\frac{1}{2}\sqrt{e^{2x}+2+e^{-2x}}\,dx$

$=\int_0^3\frac{1}{2}\sqrt{(e^x+e^{-x})^2}\,dx=\int_0^3\frac{1}{2}(e^x+e^{-x})\,dx$

$=\frac{1}{2}(e^x-e^{-x})\Big|_0^3$

$=\frac{1}{2}(e^3-e^{-3})$ ~ 11

3. The length of the curve $y=\ln(\sec x)$ from $x=0$ to $x=b$ may be represented by which of the following integrals?

$y'=\frac{\sec x\tan x}{\sec x}$

(a) $\int_0^b \sec x\,dx$ (circled)

(b) $\int_0^b \sec^2 x\,dx$

(c) $\int_0^b\left(\sqrt{1+\sec^2 x\tan^2 x}\right)dx$

$\int_0^b\sqrt{1+\tan^2 x}=\int_0^b\sqrt{\sec^2 x}\,dx=\int_0^b\sec x\,dx$

4. Using your calculator, find the length of the curve $y=x^{3/2}$ from (1,1) to (4,8).

$y'=\frac{3}{2}\sqrt{x}$

$L=\int_1^4\sqrt{1+\frac{9}{4}x}\,dx=\boxed{7.634}$

PTFs #BC 06 – Euler's Method

1. Start at the given point on the graph.
2. Calculate the slope at this point by using the differential equation.
3. For a given value of Δx, calculate the value of Δy, using the equation $\Delta y = \frac{dy}{dx}\Delta x$.
4. Add Δx to x and Δy to y to get a new point.
5. Repeat the process until you reach your desired x value.

---

1. For $\frac{dy}{dx} = \frac{-x}{2y}$, use Euler's Method starting at $f(0) = 3$ with three steps to approximate $f(1.5)$.

$y(0.5) = 3 + 0.5(0) = 3$

$y(1) = 3 + 0.5(-\frac{1}{12}) = 2.958\overline{3}$

$y(1.5) = 2.958 + 0.5(.479) = \boxed{2.21875}$

2. Let $\frac{dy}{dx} = \frac{\cos x - 1}{\sin y}$. If (1, 1) is on the graph of y, what is the value of y when $x = 1.02$ using Euler's method? Let $\Delta x = 0.01$.

   a. 0.977
   b. 0.989 (circled)
   c. 0.984
   d. 0.994
   e. 1.005

$y(1.01) = 1 + 0.01(-.546) = 0.9945$

$y(1.02) = .9945 + .01(-.558) = 0.9889539637$

If y is a differentiable function of t such that $\frac{dP}{dt} = kP(L-P)$, then

- $P = \frac{L}{1+Ce^{-Lkt}}$
- $\lim_{t\to\infty} P(t) = L$ (L is the carrying capacity.)
- $\frac{dP}{dt}$ is at its maximum (the rate of growth is increasing the fastest) when the function reaches half its carrying capacity, $\frac{L}{2}$. (this is also the point of inflection on $P(t)$)

---

1. The population, P of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where $P(0) = 3000$ and t is the time in years. What is $\lim_{t\to\infty} P(t)$?

$\frac{dP}{dt} = 2P\left(1 - \frac{P}{10000}\right)$

$\lim_{t\to\infty} P(t) = \boxed{10,000}$

2. A rumor spreads at the rate $\frac{dP}{dt} = 2P(1-P)$ where P is the portion of the population that has heard the rumor at time t. What portion of the population has heard the rumor when it is spreading the fastest?

Since the carrying capacity is 1, $\frac{dP}{dt}$ is a max at $\frac{1}{2}$

$\frac{1}{2}$ the population has heard the rumor.

- Position Vector: $r(t) = \langle x(t), y(t) \rangle$
- Velocity Vector: $v(t) = \langle x'(t), y'(t) \rangle$
- Acceleration Vector: $a(t) = \langle x''(t), y''(t) \rangle$
- Speed (magnitude of velocity vector): speed $= \sqrt{(x'(t))^2 + (y'(t))^2}$
- Total distance traveled (arc length): distance $= \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$
- Slope of the curve: $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$
- Second derivative: $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} = \frac{\left[\frac{dy}{dx}\right]'}{x'(t)}$

---

1. The position of a particle is given by $x = \sin t$ and $y = \cos(2t)$ on the interval $[0, 2\pi]$.

   a. Find the velocity vector.

   $v(t) = \langle \cos t, -2\sin 2t \rangle$

   b. For what values of t is the particle at rest?

   $\cos t = 0$, $t = \frac{\pi}{2}, \frac{3\pi}{2}$

   $-2\sin 2t = 0$: $2t = 0$, $2t = \pi$, $2t = 2\pi$, $2t = 3\pi$; $t = 0$, $t = \frac{\pi}{2}$, $t = \pi$, $t = \frac{3\pi}{2}$

   Particle at rest when it is not moving in the x or y-direction.

   c. Write the rectangular equation of the path of the particle in x and y only.

   $t = \sin^{-1} x$

   $y = \cos(2\sin^{-1} x)$

2. The velocity of a roller coaster at time t seconds can be modeled parametrically by $x'(t) = 10 + 4\cos t$ and $y'(t) = (20 - t)\sin t + \cos t - 1$ for $0 \le t \le 10$ seconds. At time $t = 0$ the rollercoaster is 1 meter high.

   a. Find the time t at which the car is at its max height.

   $y'(t) = 0$
   $t = 3.024$
   $t = 6.283$
   $t = 9.239$

   $t = 9.239$ since $\int_0^{9.239} y'(t)\,dt$ is the largest value

   b. Find the speed, in m/sec, of the car at this time.

   Speed $= \sqrt{(10 + 4\cos 9.239)^2 + (20 - 9.239\sin 9.239 + \cos 9.239 - 1)^2}$

   $= 6.069$ m/s

Polar-Rectangular Conversion Formulas:

$x = r\cos\theta$ $\qquad$ $r = \sqrt{x^2 + y^2}$

$y = r\sin\theta$ $\qquad$ $\tan\theta = \dfrac{y}{x}$

---

1. Graph the point $P(2,2)$ on a coordinate plane. Convert P to a polar coordinate and graph the polar coordinate on a different set of axes.

2. Convert $r = 9\sin\theta$ to a rectangular equation.

3. Show that the polar equation $r^2 = \dfrac{1}{\cos^2\theta - \sin^2\theta}$ can be written in rectangular form as the equation $x^2 - y^2 = 1$.

4. A curve drawn in the xy-plane is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$. Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

PTFs #BC 10 – Slopes of Polar Curves

The slope of a polar curve is still the slope of the tangent line:

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]}$$

Horizontal tangents when $\frac{dy}{d\theta}=0$.

Vertical tangents when $\frac{dx}{d\theta}=0$.

No conclusion if both $\frac{dy}{d\theta}=0$ and $\frac{dx}{d\theta}=0$.

---

1. Find the points where the horizontal and vertical tangent lines occur for $r=\sin\theta$ for $0\le\theta\le\pi$.

$y = r\sin\theta$

$y = \sin\theta\sin\theta$

$y = \sin^2\theta$

$\frac{dy}{d\theta} = 2\sin\theta\cos\theta = 0$

$\boxed{\theta = 0, \frac{\pi}{2}, \pi}$

2. Consider the polar curve $r=2\sin(3\theta)$ for $0\le\theta\le\pi$. Find the slope of the curve at the point where $\theta=\frac{\pi}{4}$.

$r = 2\sin 3\theta$

$r' = 6\cos 3\theta$

$r(\pi/4) = \sqrt{2}$

$r'(\pi/4) = -3\sqrt{2}$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{4}} = \frac{-3\sqrt{2}\cdot\frac{\sqrt{2}}{2} + \sqrt{2}\left(-\frac{\sqrt{2}}{2}\right)}{-3\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) - \sqrt{2}\left(\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{-3 \quad -1}{3 \quad -1}$$

$$= \frac{-4}{2} = \boxed{-2}$$

The area of a polar region is:

$$A=\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

1. Find the area inside the curve of $r=3\cos(3\theta)$ for $0\le\theta\le\pi$.

$A = 3\cdot\int_0^{\pi/3}(3\cos 3\theta)^2 d\theta$

$= 14.137$

2. The area of the region bounded by the polar graph of $r=\sqrt{3+\cos\theta}$ is given by the integral:

a. $\int_0^{2\pi}\left(\sqrt{3+\cos\theta}\right)d\theta$

b. $\int_0^{\pi}\left(\sqrt{3+\cos\theta}\right)d\theta$

c. $2\int_0^{\pi/2}(3+\cos\theta)d\theta$

(d.) $\int_0^{\pi}(3+\cos\theta)d\theta$

e. $2\int_0^{\pi/2}\left(\sqrt{3+\cos\theta}\right)d\theta$

3. Find the area of the region that is inside $r=4-4\cos\theta$ and outside of $r=6$.

$4-4\cos\theta=6$

$-4\cos\theta=2$

$\cos\theta=-\frac{1}{2}$

$\theta=\frac{2\pi}{3},\frac{4\pi}{3}$

$A=\int_{\frac{2\pi}{3}}^{\pi}(4-4\cos\theta)^2-(6)^2\,d\theta$

$= 18.611$

PTFs #BC 12 – Polar Graphs and Motion

Variable and its direction of movement:

R = radius $\frac{dr}{dt}$ measures movement going toward/away from the origin

X = horizontal distance $\frac{dx}{dt}$ measures movement going toward/away from y-axis

Y = vertical distance $\frac{dy}{dt}$ measures movement going toward/away from x-axis

You must check both the location and the direction of movement!

Problems work just like vectors, just with trig functions. All vector formulas apply.

A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. (Calc.)

a) Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

$x = r\cos\theta$

$x = (4-2\sin\theta)\cos\theta$

$-1 = (4-2\sin\theta)\cos\theta$

$-1 = (4-2\sin(t^2))\cos(t^2)$

$t = 1.428$

b) Is the particle moving towards or away from the y-axis at the time found in part (a)? Show the analysis that leads to your answer.

$y = r\sin\theta$

$y = (4-2\sin\theta)\sin\theta$

$y = (4-2\sin t^2)\sin t^2$

$\frac{dy}{dt} = (-4t\cos(t^2))\sin(t^2) + (4-2\sin(t^2))(2t\cos(t^2))$

$\frac{dy}{dt}\Big|_{t=1.428} = -0.555 < 0$

moving towards the y-axis

c) Find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

$\langle 4\cos(t^2) - 2\sin(t^2)\cos(t^2),\ 4\sin(t^2) - 2\sin^2(t^2) \rangle$

$V(1.5) = \langle -8.072, -1.673 \rangle$

A Power Series can be used to represent a function, but only on a specified domain or interval. It involves a variable, normally x, instead of just constants.

- A Power series centered at $x=0$ is of the form $\sum_{n=1}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$.
- A Power series centered at $x=c$ is of the form $\sum_{n=1}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$.

Find the first four non-zero terms and the general terms of the series.

1. $\frac{1}{1-x}$ centered at $x=0$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$= 1 + x + x^2 + x^3 + \dots x^n$

2. $\frac{2}{x-1}$ centered at $x=0$

$\frac{2}{x-1} = \frac{2}{-1+x} = \frac{-2}{1-x} = -2\sum_{n=0}^{\infty} x^n$

$\frac{2}{x-1} = -2 - 2x - 2x^2 - 2x^3 - \dots -2x^n$

3. $\frac{6}{4+2x}$ centered at $x=1$

$\frac{6}{4+2x} = 6 \cdot \frac{1}{4+2x} = \frac{6}{4} \cdot \frac{1}{1+(\frac{1}{2}x)} = \frac{6}{4}\sum_{n=0}^{\infty} (\frac{1}{2}x - 1)^n$

$\frac{6}{4+2x} = \frac{6}{4} + \frac{6}{4}(\frac{1}{2}x-1) + \frac{6}{4}(\frac{1}{2}x-1)^2 + \frac{6}{4}(\frac{1}{2}x-1)^3 + \dots$

$+ \frac{6}{4}(\frac{1}{2}x-1)^n$

The **Taylor Polynomial** of order n for f centered at $x=c$ is given by:

$$P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

The **Maclaurin Polynomial** of order n for f centered at $x=0$ is given by:

$$P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n$$

To construct a Taylor polynomial or series:

1. Evaluate $f(x)$ and it's first n derivatives at $x=c$.
2. Plug into one of the formulas above to write the series/polynomial.
3. If it is a series, make sure to include the . . . and to write the general term.

---

1. Find the first four terms of the Taylor polynomial for $f(x)=\frac{1}{x-1}$ about $x=2$.

$f(x)=\frac{1}{x-1}$ $f(2)=1$

$f'(x)=\frac{-1}{(x-1)^2}$ $f'(2)=-1$

$f''(x)=\frac{2}{(x-1)^3}$ $f''(2)=2$

$f'''(x)=-\frac{6}{(x-1)^4}$ $f'''(2)=-6$

$$\frac{1}{x-1}=1-(x-2)+\frac{2(x-2)^2}{2!}-\frac{6(x-2)^3}{6!}+\ldots$$

2. Let f be a function and $f(2)=-3$, $f'(2)=5$, $f''(2)=3$ and $f'''(2)=-8$. Write a third degree Taylor polynomial for f about $x=2$ and use it to approximate $f(1.5)$.

$$T_3(x)=-3+5(x-2)+\frac{3(x-2)^2}{2!}-\frac{8(x-2)^3}{3!}$$

$$T_3(1.5)=-3+5(-0.5)+\frac{3(-0.5)^2}{2}-\frac{8(-0.5)^3}{6}$$

$T_3(1.5)=\underline{-4.958}$

3. Find the fourth-degree Taylor polynomial for $f(x)=\ln(x)$ about $x=1$ and use it to approximate the value of $\ln(1.1)$.

$f(x)=\ln x$ $f(1)=0$

$f'(x)=\frac{1}{x}$ $f'(1)=1$

$f''(x)=-\frac{1}{x^2}$ $f''(1)=-1$

$f'''(x)=\frac{2}{x^3}$ $f'''(1)=2$

$$T_4(x)=0+(x-1)-\frac{(x-1)^2}{2!}+\frac{2(x-1)^3}{3!}-\frac{6(x-1)^4}{4!}+\ldots$$

$$T_4(1.1)=0.1-\frac{(0.1)^2}{2}+\frac{2(0.1)^3}{6}-\frac{6(0.1)^4}{24}$$

$=\underline{0.095}$

The **Taylor Series** generated by f at $x=c$ is defined by:

$$f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

The **Maclaurin Series** generated by f at $x=c$ is defined by:

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

(*This is just a Taylor series that is centered at $x=0$!)

To construct a Taylor polynomial or series:

4. Evaluate $f(x)$ and it's first n derivatives at $x=c$.
5. Plug into one of the formulas above to write the series/polynomial.
6. If it is a series, make sure to include the . . . and to write the general term.

---

1. The Taylor series about $x=5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence of f, and $f(5)=0$. The nth derivative of f at $x=5$ is given by $f^{(n)}(5)=\dfrac{(-1)^n n!}{2^n(n+2)}$. Write the first four terms and the general term for the Taylor series about $x=5$.

$f(5)=\frac{1}{2}$

$f'(5)=\frac{-1}{6}$

$f''(5)=\frac{2}{16}=\frac{1}{8}$

$f'''(5)=\frac{-6}{40}=-\frac{3}{20}$

$f^{4}(5)=\frac{24}{96}=\frac{1}{4}$

$T(x)=\frac{1}{2}-\frac{1}{6}(x-5)+\frac{\frac{1}{8}(x-5)^2}{2!}-\frac{\frac{3}{20}(x-5)^3}{3!}+\frac{\frac{1}{4}(x-5)^4}{4!}+\ldots$

$+\frac{(-1)^n n!}{2^n(n+2)}\cdot\frac{(x-5)^n}{n!}$

Simplifies: $\frac{(-1)(x-5)^n}{2^n(n+2)}$

2. The Taylor series about $x=0$ for f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x=0$ is given by $f^{(n)}(0)=\dfrac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2}$ for $n\geq 2$. The graph of f has a horizontal tangent line at $x=0$, and $f(0)=6$. Write the first four terms of the Maclaurin series for f about $x=0$.

$f(0)=6$

$f'(0)=0$

$f''(0)=\frac{-6}{25}$

$f'''(0)=\frac{24}{500}=\frac{6}{125}$

$f^{4}(0)=\frac{-120}{5625}=-\frac{8}{375}$

$T_4(x)=6-\frac{\frac{6}{25}x^2}{2!}+\frac{\frac{6}{125}x^3}{3!}-\frac{\frac{8}{375}x^4}{4!}+\ldots$

PTFs #BC 16 – Algebraic Manipulation of Series

Power series that you must know:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

You may manipulate the series above to find new series (as long as you are centered at $x=0$) by:

1. Substituting into a know series.
2. Multiplying/dividing by a constant and/or variable.
3. Adding/subtracting two known series.

---

1. Let f be the function given by $f(x) = 6e^{-x/3}$. Find the first four non-zero terms and the general term for the Taylor series for f about $x=0$.

$6e^{-x/3} = 6\sum_{n=0}^{\infty} \frac{(-\frac{x}{3})^n}{n!}$

$= 6 - 2x + \frac{2x^2}{3} - x^3 + \ldots \frac{6(-x/3)^n}{n!}$

2. Write a power series and the general term for $\sin(2x)$.

$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$

$= (2x) - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \ldots \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$

3. Find the first three non-zero terms and the general term of the Taylor polynomial for $g(x) = \frac{\cos(x^2) - 1}{x}$ centered at $x=0$.

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots$

$\cos(x^2) - 1 = -\frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots$

$\frac{\cos(x^2) - 1}{x} = -\frac{x^3}{2!} + \frac{x^7}{4!} - \frac{x^{12}}{6}$

PTFs #BC 17 – Calculus Manipulation of Series

You may manipulate series to find new series (as long as you are centered at $x=0$) by:

1. Differentiating a known series.
2. Integrating a known series.

---

1. Let $P(x)=7-3(x-4)+5(x-4)^2-2(x-4)^3+6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives for all orders of all real numbers.

   a. Write the second degree Taylor polynomial for $f'(x)$ about 4 and use it to approximate $f'(4.3)$.

$f'(x) = -3x + 10(x-4) - 6(x-4)^2$

$f'(4.3) = -3(4.3) + 10(0.3) - 6(0.3)^2$

$= -10.44$

   b. Write a fourth-degree Taylor polynomial for $g(x)=\int_4^x f(t)dt$.

$\int_4^x 7-3(t-4)+5(t-4)^2-2(t-4)^3 dt=$

$7t - \frac{3(t-4)^2}{2} + \frac{5(t-4)^3}{3} - \frac{2(t-4)^4}{4} \Big|_4^x$

$7x - \frac{3(x-4)^2}{2} + \frac{5(x-4)^3}{3} - \frac{2(x-4)^4}{4} - 28$

$g(x) = 7(x-4) - \frac{3(x-4)^2}{2} + \frac{5(x-4)^3}{3} - \frac{(x-4)^4}{2}$

2. The Maclaurin series for the function f is given by $f(x)=\sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1}=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{1296}{4}x^4+\ldots$ on its interval of convergence. Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

$f'(x) = 2 + 4x + 8x^2 + 1116x^3 + \ldots + \frac{(n+1)x^n \cdot 2^{n+1}}{(n+1)}$

$x^n 2^{n+1}$

3. For $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-5)^n}{n\cdot 5^n}$

   a. Find the power series for $f'(x)$.

$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n(x-5)^{n-1}}{n\, 5^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^{n-1}}{5^n}$

   b. Find the power series for $\int f(x)dx$.

$\int f(x)\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \frac{(x-5)^{n+1}}{n+1}}{n\, 5^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^{n+1}}{n(n+1)\, 5^n}$

- Geometric Series: $a + ar + ar^2 + ar^3 + \ldots = \sum_{n=1}^{\infty} a_1 r^{n-1}$
  - A geometric series with common ration, r will:
    - Converge if $|r| < 1$ (Sum: $S = \frac{a_1}{1-r}$)
    - Diverge if $|r| > 1$

For #1-3, determine if each sum diverges or converges. If it converges, find the sum.

1. $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^{n-1}$ $= \frac{3}{1-\frac{1}{2}} = \boxed{6}$

2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots + \left(-\frac{1}{2}\right)^{n-1} + \ldots$ $= \frac{1}{1-(-\frac{1}{2})}$

   $= \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}$

3. $\frac{\pi}{2} + \frac{\pi^2}{4} + \frac{\pi^3}{8} + \ldots$ $= \frac{\pi/2}{1-\pi/2} = \frac{\pi/2}{\frac{2-\pi}{2}} = \boxed{\frac{\pi}{2-\pi}}$

Find the sum.

4. $\sum_{i=n}^{\infty} \left(\frac{1}{3}\right)^i$ $= \frac{(1/3)^n}{1-1/3} = \frac{(1/3)^n}{2/3} = \frac{3}{2}\left(\frac{1}{3}\right)^n$

(A) $\frac{3}{2} - \left(\frac{1}{3}\right)^n$

(B) $\frac{3}{2}\left[1 - \left(\frac{1}{3}\right)^n\right]$

(C) $\frac{3}{2}\left(\frac{1}{3}\right)^n$ (circled)

(D) $\frac{2}{3}\left(\frac{1}{3}\right)^n$

(E) $\frac{2}{3}\left(\frac{1}{3}\right)^{n+1}$

PTFs #BC 19 – nth Term Test

- If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- This test is a test for divergence only! It can never be used to prove convergence.

Determine if each series diverges or if the nth term test is inconclusive.

1. $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n-5}\right)$

$\lim_{n\to\infty} \frac{2n+3}{3n-5} = \frac{2}{3}$

Diverges

2. $\sum_{n=1}^{\infty}\frac{3^n-2}{3^n}$

$\lim_{n\to\infty} \frac{3^n-2}{3^n} = \lim_{n\to\infty} \frac{3^n}{3^n} - \frac{2}{3^n}$

$= \lim_{n\to\infty} 1 - \frac{2}{3^n} = 1 - 0 = 1$

Diverges

3. $\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{3}\right)\right)^n$

$\lim_{n\to\infty} \cos\left(\frac{\pi}{3}\right)^n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

Inconclusive

* since $\left|\cos\left(\frac{\pi}{3}\right)\right| < 1$, $\sum_{n=1}^{\infty}\left(\cos\left(\frac{\pi}{3}\right)\right)^n$ converges by geometric test

4. $\sum_{n=0}^{\infty}\left(\frac{n+3}{n^2-9n-10}\right)$

$\lim_{n\to\infty} \frac{n+3}{n^2-9n-10} = 0$

Inconclusive

* Diverges by Comparison / P-series Tests

A series of the form $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ is an alternating series whose terms alternate pos/neg or neg/pos and where $a_n > 0$. The series will converge if both conditions are met:

- $\lim_{n\to\infty} a_n = 0$ (the limit of the terms is going to zero)
- $|a_{n+1}| \le |a_n|$ (the terms are decreasing in magnitude)

Determine if each series converge or diverge.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\frac{1}{n}$ is decreasing

Converges

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2n}{4n-3}\right)$

$\lim_{n\to\infty} \frac{2n}{4n-3} = \frac{1}{2}$

Diverges by n^{th} term test

3. Which of the following series converge? (You may use any of the tests to decide.)

I. $\sum_{n=1}^{\infty}\frac{n}{n+2}$

II. $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

III. $\sum_{n=1}^{\infty}\frac{1}{n}$

(A) None (B) II (C) III

(D) I and II (E) I and III

$\sum \frac{n}{n+2}$ $\lim_{n\to\infty} \frac{n}{n+2} = 1$ Diverges (p-series)

$\sum \frac{\cos(n\pi)}{n} = \frac{-1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots = \sum \frac{(-1)^n}{n}$ Converges (AST)

$\sum \frac{1}{n}$ Diverges

If f is continuous, positive and decreasing for $x \geq 1$ and $a_n = f(x)$ then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

2. State that the function $a_n = f(x)$ is continuous and positive and then use $f'(x)$ to show that the function is decreasing.
3. Evaluate $\int_1^{\infty} f(x)dx$.
   - If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.
   - If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.
4. The sum, S, can be approximated by finding the nth partial sum, S_n and then finding the remainder, R_n. $S = S_n + R_n$ where $0 \leq R_n \leq \int_n^{\infty} f(x)dx$.

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} ne^{-n^2}$

$\lim_{b\to\infty} \int_1^b xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}\Big|_1^b = -\frac{1}{2}e^{-b^2} + \frac{1}{2}e^{-1}$

$\lim_{b\to\infty} -\frac{1}{2e^{b^2}} + \frac{1}{2e} = 0 + \frac{1}{2e}$

Converges

2. $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

$\lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}dx = \lim_{b\to\infty} \int_2^b \frac{1}{u}du = \lim_{b\to\infty} \ln|u|\Big|_2^b$

$u = \ln x$

$du = \frac{1}{x}dx$

$x\,du = dx$

$= \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 2|$

$= \infty - \ln|\ln 2|$

Diverges

3. Let f be a positive, continuous, decreasing function such that $a_n = f(n)$. If $\sum_{n=1}^{\infty} a_n$ converges to k, which of the following must be true?

(A) $\lim_{n\to\infty} a_n = k$

(B) $\int_1^n f(x)dx = k$

(C) $\int_1^{\infty} f(x)dx$ diverges

(D) $\int_1^{\infty} f(x)dx$ converges

(E) $\int_1^{\infty} f(x)dx = k$

PTFs #BC 22 – p-Series Test

A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+...$ is a p-Series, where p is a positive constant.

- The series converges if $p>1$.
- The series diverges if $p\leq 1$.

(Remember the graph of $f(x)=\frac{1}{x}$ and how it diverged because it was "too far away from the x- and y-axis." If the exponent is 1 or smaller, it will pull the graph further away and the series will not diverge. If the exponent is greater than 1, the terms are smaller and the graph is much closer to the axes; these series converge.)

Determine if each series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$

Converges $p=2$

2. $\sum_{n=1}^{\infty}\frac{5}{\sqrt{n}}$

Diverges $p=\frac{1}{2}$

3. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2}}$

Diverges $p=\frac{2}{3}$

Ratio test is useful for series that converge rapidly such as exponential functions and factorials.

- A series converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$.
- A series diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ (or $=\infty$).
- The ratio test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$.

*This test is essentially finding the ratio of two adjacent terms. It is similar to finding the ratio of a geometric series and so has the same convergence/divergence specifications.

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{3^n}{n^2}\right)$

$$\lim_{n\to\infty}\left|\frac{3^{n+1}}{3^n}\cdot\frac{n^2}{(n+1)^2}\right| = \lim_{n\to\infty}\frac{3^n\cdot 3\, n^2}{3^n\,(n+1)^2}$$

$$= 3\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 3\cdot 1 = 3$$

Diverges

2. $\sum_{n=0}^{\infty}\frac{2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\frac{2^n\cdot 2\, n!}{2^n\,(n+1)\,n!}$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0$$

Converges

Direct comparison test is useful to compare series that are similar to known convergent or divergent series.

5. Choose an appropriate "parent", b_n, series (usually a geometric or p-series).
6. Determine whether the "parent", b_n, series converges or diverges.
7. Show that your series fits one of the two statements hold true:
   - If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. (Less than a convergent series converges.)
   - If $0 \le b_n \le a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges. (Greater than a divergent series diverges.)

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{1}{n^3+1}\right)$

$\sum_{n=1}^{\infty} \frac{1}{n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^3}$

Converges since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series

2. $\sum_{n=1}^{\infty} 2^{-n!}$

$\sum_{n=1}^{\infty} \frac{1}{2^{n!}} < \sum_{n=1}^{\infty} \frac{1}{2^n}$

Converges since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by geometric series test

3. $\sum_{n=0}^{\infty} \frac{1}{3^n+2}$

$\sum_{n=0}^{\infty} \frac{1}{3^n+2} < \sum_{n=0}^{\infty} \frac{1}{3^n}$

Converges since $\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges by geometric series test

4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-2}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-2} > \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Diverges since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series test

This test compares "messy" algebraic series in question, a_n, with a known similar convergent or divergent series, b_n, by taking the quotient of the two series.

8. Choose an appropriate "parent", b_n, series (usually a geometric or p-series).
9. Determine whether the "parent", b_n, series converges or diverges.
10. Set up $\lim_{n\to\infty}\frac{a_n}{b_n}$ and evaluate to find the limit, L.
11. Show that your limit L, fits one of the three statements below:
    - If L is finite and positive, then $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.
    - If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
    - If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}\left(\frac{1}{3n^2+4n+5}\right)$

$b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{\frac{1}{3n^2+4n+5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^2+4n+5} = 3$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series, then $\sum_{n=1}^{\infty} \frac{1}{3n^2+4n+5}$ converges

2. $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$

$b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{1/n} = \lim_{n\to\infty} n\sin(\frac{1}{n}) = \infty$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ diverges

3. $\sum_{n=0}^{\infty}\frac{n^2-10}{4n^5+n^3-2}$

$b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{n^2-10}{4n^5+n^3-2}\cdot\frac{1}{1/n^2} = \lim_{n\to\infty} \frac{n^4-10n^2}{4n^5+n^3-2} = 0$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then $\sum_{n=1}^{\infty} \frac{n^2-10}{4n^5+n^3-2}$ converges

4. $\sum_{n=1}^{\infty}\frac{1}{n^3-2}$

$b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{1/n^3-2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^3-2} = 0$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then $\sum_{n=1}^{\infty} \frac{1}{n^3-2}$ converges

PTFs #BC 26 – Radius and Interval of Convergence

We call R the **radius of convergence**, which is how far away from the center the series will converge. The convergence of a power series has one of three possibilities:

1. The series converges only at $x = c$ so $R = 0$.
2. The series converges for all values of x so $R = \infty$.
3. There exists an $R > 0$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$.

To find the radius and interval of convergence:

7. Use the Geometric Series or the Ratio test to set up an absolute value inequality.
8. Solve the absolute value inequality - this will give you your radius of convergence.
9. Test the endpoints to determine convergence and write your interval of convergence.

Find the radius and interval of convergence for the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n \cdot 2^n}$

$\displaystyle\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(x-2)^n}\frac{n2^n}{(n+1)2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)(x-2)^n\, n2^n}{(x-2)^n(n+1)2\cdot2^n}\right|$

$\displaystyle\lim_{n\to\infty}|x-2|\frac{n}{2(n+1)} = \frac{1}{2}|x-2| < 1$

$|x-2| < 2$

$-2 < x-2 < 2$

$0 < x < 4$

$R = 2$

$0 \le x \le 4$

$x=0$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+2}2^n}{n2^n}$ converges by AST

$x=4$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n}{n2^n}$ converges by AST

2. $f(x) = \dfrac{1}{3} + \dfrac{2}{3^2}x + \dfrac{3}{3^3}x^2 + \ldots = \displaystyle\sum_{n=0}^{\infty}\frac{n+1}{3^{n+1}}x^n$

$\displaystyle\lim_{n\to\infty}\left|\frac{n+2}{n+1}\frac{x^{n+1}}{x^n}\frac{3^{n+1}}{3^{n+2}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^n x\, 3^{n+1}}{(n+1)x^n\, 3\cdot 3^{n+1}}\right|$

$= \displaystyle\lim_{n\to\infty}|x|\frac{n+2}{3(n+1)} = \frac{1}{3}|x| < 1$

$|x| < 3$

$-3 < x < 3$

$R = 3$

$-3 < x < 3$

$x=3$: $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{3^{n+1}}3^n = \sum_{n=0}^{\infty}\frac{n+1}{3}$ diverges

$x=-3$: $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{3^{n+1}}(-3)^n = \sum_{n=0}^{\infty}(-1)^n\frac{n+1}{3}$ diverges

3. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$

$\displaystyle\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\frac{n}{(n+1)}\right| = \lim_{n\to\infty}|x|\frac{n}{n+1} = |x| < 1$

$-1 < x < 1$

$x=1$: $\displaystyle\sum_{n=1}^{\infty}\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges

$x=-1$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by AST

$R = 1$

$-1 \le x < 1$

4. $\displaystyle\sum_{n=0}^{\infty}\frac{10x^n}{n!}$

$\displaystyle\lim_{n\to\infty}\left|\frac{10x^{n+1}}{10x^n}\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{10x^n x\, n!}{10x^n (n+1)n!}\right| = \lim_{n\to\infty}|x|\frac{1}{n+1} = 0$

converges for all x

$-\infty < x < \infty$

PTFs #BC 27 – Alternating Series Remainder

If an alternating series converges, then the sum of the series, S can be approximated by finding the nth partial sum, S_n. The partial sum is within R_n, the remainder, of the actual sum. R_n is less than or equal to the first neglected term.

$$|R_n| = |S - S_n| \le a_{n+1}$$

1. For the series $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n}\right)$, find the error in using the first 100 terms to estimate the sum.

$R_{100} < a_{101} = (-1)^{102}\frac{1}{101} = \frac{1}{101}$

Error $< \frac{1}{101}$

2. Estimate the amount of error involved in the approximation of $f(-1)$ for the first three terms of $f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{3^n n!}$.

$R_3 < a_4 = \frac{(-1)^4 4^4}{3^4 4!} = \frac{4\cdot4\cdot4\cdot4}{3\cdot3\cdot3\cdot3\cdot4\cdot3!} = \frac{64}{486} = \frac{32}{243}$

$R_3 < \frac{32}{243}$

3. Determine the number of terms required to approximate the sum of the convergent series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} = \frac{1}{\sqrt{e}}$ with an error of less than 0.001.

$a_0 = 1$

$a_1 = \frac{-1}{2}$

$a_2 = \frac{1}{8}$

$a_3 = \frac{-1}{48} = 0.0208$

$a_4 = \frac{1}{384} = 0.0026$

$a_5 = \frac{-1}{3840} = 0.00026 \Rightarrow$ need first 5 terms to get an error less than 0.001

PTFs #BC 28 – Lagrange Error Bound

A function $f(x)$ is approximated by the sum of a Taylor polynomial $P_n(x)$ and some remainder $R_n(x)$, given by $f(x)=P_n(x)+R_n(x)$.

Therefore the error bound associated with the remainder is defined as:

Error = $|R_n(x)|=|f(x)-P_n(x)|$ with $|R_n(x)|\le\left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{(n+1)}\right|$ where $f^{(n+1)}(z)$ is the max value of the derivative on the specified interval.

- Remember, it is just the first term left off with the max value of the derivative used.
- If the max derivative is given, use it!
- If not, find the next derivative and look for max spot to evaluate.

---

1. Let f be the function given by $g(x)=\sin\left(5x+\frac{\pi}{4}\right)$ and let $P(x)$ be the third-degree Taylor polynomial for f about $x=0$. Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right|<\frac{1}{100}$.

$x=\frac{1}{10}$

$P_2(x)=(5x+\frac{\pi}{4})-\frac{(5x+\frac{\pi}{4})^3}{3!}$

$g'(x)=5\cos(5x+\frac{\pi}{4})$

$g''(x)=-25\sin(5x+\frac{\pi}{4})$

$g'''(x)=-125\cos(5x+\frac{\pi}{4})$

$g^4(x)=625\sin(5x+\frac{\pi}{4})$

$R_2\le\frac{625}{(3+1)!}(\frac{1}{10}-\frac{\pi}{20})^{3+1}$

$\le\frac{625}{24}(\frac{2-\pi}{20})^4$

$\le 0.0002764 < .01$

2. Let f be a function that has derivatives of all orders for $(2.5,3.5)$. Assume that $f(3)=1$, $f'(3)=-3$, $f''(3)=12$, and $|f'''(x)|\le 36$ for all x in $(2.5,3.5)$.

   a. Find the second-order Taylor polynomial about $x=3$ for $f(x)$.

   $P_2(x)=1-3(x-3)+\frac{12(x-3)^2}{2!}$

   b. Estimate $f(2.7)$ using part (a).

   $P_2(2.7)=1-3(-0.3)+6(-0.3)^2$

   $P_2(2.7)=2.44$

   c. What is the max possible error in estimating part (b)?

   $R_2\le\left|\frac{36}{3!}(2.7-3)^3\right|=6(0.3)^3=0.162$

   $R_2\le 0.162$