12.3 Tangent Lines and Velocity

1 Tangent Lines In Lesson 1-4, you calculated the average rate of change between two points on the graph of a nonlinear function by finding the slope of the secant line through these points. In this lesson, we develop a way to find the slope of such functions at one instant or point on the graph.

The graphs below show successively better approximations of the slope of $y = x^2$ at (1, 1) using secant lines.

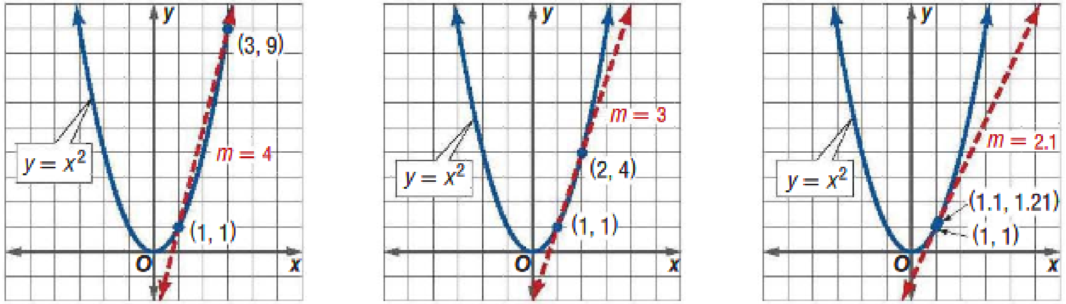

Figure 12.3.1 **Figure 12.3.2** **Figure 12.3.3**

Notice as the rightmost point moves closer and closer to (1, 1), the secant line provides a better linear approximation of the curve near that point. We call the best of these linear approximations the **tangent line** to the graph at (1, 1). The slope of this line represents the rate of change in the slope of the curve at that instant. To define each of these terms more precisely, we use limits.

To define the slope of the tangent line to $y = f(x)$ at the point $(x, f(x))$, find the slope of the secant line through this point and one other point on the curve. Let the x-coordinate of the second point be $x + h$ for some small value of h. The corresponding y-coordinate for this point is then $f(x + h)$, as shown in Figure 12.3.4. The slope of the secant line through these two point is given by

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} \text{ or } \frac{f(x+h) - f(x)}{h}.$$

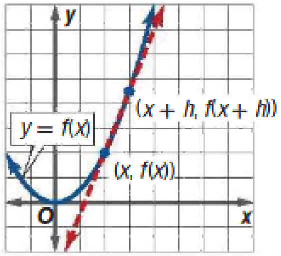

Figure 12.3.4

This expression is called the **difference quotient**.

As the second point approaches the first, or as $h \to 0$, the secant line approaches the tangent line at $(x, f(x))$. We define the slope of the tangent line at x, which represents the instantaneous rate of change of the function at that point, by finding the limits of the slopes of the secant lines as $h \to 0$.

KeyConcept Instantaneous Rate of Change

The instantaneous rate of change of the graph of $f(x)$ at the point $(x, f(x))$ is the slope m of the tangent line at $(x, f(x))$ given by $m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.

Find the slope of the line tangent to the graph of $y = x^2$ at (1, 1).

Find the slope of the line tangent to the graph of each function at the given point.

1A. $y = x^2$; (3, 9)

Find an equation for the slope of the graph of $y = \frac{4}{x}$ at any point.

Find an equation for the slope of the graph m of each function at any point.

2A. $y = x^2 - 4x + 2$

2 Instantaneous Velocity In Lesson 1-4, you calculated the average speed of a dropped object by dividing the distance traveled by the time it took for the object to cover that distance. Velocity is speed with the added dimension of direction. You can calculate average velocity using the same approach that you used when calculating average speed.

KeyConcept Average Velocity

If position is given as a function of time $f(t)$, for any two points in time a and b, the average velocity v is given by

$$v_{avg} = \frac{\text{change in distance}}{\text{change in time}} = \frac{f(b) - f(a)}{b - a}.$$

MARATHON The distance in miles that a runner competing in the Boston Marathon has traveled after a certain time t in hours can be found by $f(t) = -1.3t^2 + 12t$. What was the runner's average velocity between the second and third hour of the race?

KeyConcept Instantaneous Velocity

If the distance an object travels is given as a function of time $f(t)$, then the instantaneous velocity $v(t)$ at a time t is given by

$$v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},$$

provided the limit exists.

A baseball is dropped from the top of a building 2000 feet above the ground. The height of the baseball in feet after t seconds is given by $f(t) = 2000 - 16t^2$. Find the instantaneous velocity $v(t)$ of the baseball at 5 seconds.

The distance a particle moves along a path is given by $s(t) = 18t - 3t^3 - 1$, where t is given in seconds and the distance of the particle from its starting point is given in centimeters. Find the equation for the instantaneous velocity $v(t)$ of the particle at any point in time.

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