Pre-Calculus

A+ COLLEGE READY

Module 1
1st Nine Weeks

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Window Pane Graphing of Trigonometric Functions

Graph one cycle of the following trigonometric functions on the coordinate axes.

1. \( f(x) = -2 \sin(2(x-1)) \)
2. \( f(x) = \cos\left(\frac{1}{4}(x+1)\right) \)
3. \( f(x) = 3 \sin\left(\frac{x-\pi}{3}\right) + 1 \)
4. \( f(x) = \frac{1}{2} \sin(2x-4) + 3 \)
5. \( f(x) = 2 \cos(3(x-1)) - 4 \)

6. Draw a window pane for a tangent function and label the dimensions.

7. How can a window be used to draw the graph of \( f(x) = \csc x \)?

8. A certain chemical reaction occurs most efficiently if the temperature oscillates between a low of 35\(^\circ\)C and a high of 100\(^\circ\)C over a 15 minute interval. The temperature is at its lowest point at \( t = 0 \). Use window pane graphing to write the equation of a cosine function as temperature with time in hours.

9. The pendulum of a grandfather clock has a horizontal displacement from left to right of 18 inches. If it takes 8 seconds for the pendulum to move from right to left, use window pane graphing to write an equation for this cosine function with time in seconds.

10. An oil company is doing exploratory drilling at two locations 3 miles from town and 8 miles from town in the same direction. At the 3 mile site they discover oil at 300 ft. At the 8 mile site, they hit oil at 500 ft. If they assume the depth of the oil follows a sinusoidal path between these two depths, using the drilling depths as critical points, write an equation for the depth of the oil as a function of distance from the town; use a cosine function. (This could have multiple solutions.)
Fitting Trigonometric Models to Data

In 2000 as a Millennium celebration, the largest Ferris Wheel in the world opened in London, England, called the London Eye. It carries up to 15,000 passengers a day in 32 capsules. The following data was collected for the height above ground (in feet) of a person riding the wheel with respect to time (in minutes).

<table>
<thead>
<tr>
<th>time</th>
<th>height</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7.5</td>
<td>225</td>
</tr>
<tr>
<td>15</td>
<td>450</td>
</tr>
<tr>
<td>22.5</td>
<td>225</td>
</tr>
<tr>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>37.5</td>
<td>225</td>
</tr>
<tr>
<td>45</td>
<td>450</td>
</tr>
<tr>
<td>52.5</td>
<td>225</td>
</tr>
<tr>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>67.5</td>
<td>225</td>
</tr>
<tr>
<td>75</td>
<td>450</td>
</tr>
<tr>
<td>82.5</td>
<td>225</td>
</tr>
<tr>
<td>90</td>
<td>0</td>
</tr>
</tbody>
</table>

1. Describe the motion of the person from observing the data.

2. From the table:
   a) How many full rotations does this table show?
   b) Explain your rationale.

3. Determine the length of one period for this situation.

4. Within one period, what is the absolute maximum height of the person?
5. Graph the data on the following grid, labeling the axes.

6. Explain why the graph of the function that models this situation should not be linear.

7. Write an equation that models this particular data using \( t \) for time and \( s(t) \) for the height of the person; use a sine function.

8. Using transformations, write your equation using a cosine function.

9. If your function had “ + 4” added to the equation, how would that affect the real scenario of the person on the Ferris Wheel?

10. The company building the Ferris Wheel has decided the Ferris Wheel may run too fast and decreases the rotation speed to 40 minutes.

   a) Write a new equation giving height of a person using the sine function.

   b) Predict the position of a person on this Ferris Wheel after 8 minutes.
11. Create your own trigonometric function using a data collection device to measure a partner doing pushups where $t$ is measured in seconds and $s(t)$ is the height of the person’s chin above ground in inches.

   a) Gather the data and graph the data below to show the function is indeed a trig function.

   b) What is the period of your function? What does it mean to this problem situation?

   c) Find the first relative maximum. What factors contributed to this particular value?

12. Design another problem situation that will yield a trigonometric function when the data is collected and graphed. Use a situation not mentioned in your text and explain why you think this would be a good trigonometric model.
Linking Trigonometry and Statistics

Kenny Seymore is interested in estimating the percent of the moon that is showing. He found the following data for January and February, 2003. Help him write an equation to estimate the percent of the moon showing for any day in 2003.

1. Create a scatterplot of the data below.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage of moon showing</td>
<td>2</td>
<td>10</td>
<td>18</td>
<td>25</td>
<td>32</td>
<td>40</td>
<td>47</td>
<td>54</td>
</tr>
</tbody>
</table>

2. What transformations to a sinusoidal curve are necessary to fit a function to your graph?

3. Type the data into your calculator and adjust the values of your transformations until you have a good-fitting sinusoidal curve.

Equation of your best good-fitting sinusoidal curve: ________________________________.

4. Using your equation, what percentage of the moon do you predict to be showing on January 4th? On March 15th? On July 4th?

5. Does using this equation to extrapolate the percentages for any day cause any possible bias? Explain.

6. Perform a regression on your data using your calculator. How does your function in question 3 compare with this function? Explain.

The temperature outside a house during a 24-hour period is given by
\[ F(t) = 80 - 10\cos\left(\frac{\pi}{12}t\right), \quad 0 \leq t \leq 24, \]

Where \( F(t) \) is measured in degrees Fahrenheit and \( t \) is measured in hours.

(a) Sketch the graph of \( F \) on the grid below.

(b) What are the units of area in this example?

(c) Estimate the area under the curve with four left hand rectangles (LHR₄), then with four right hand rectangles (RHR₄), and then with four trapezoids (T₄).

(d) Describe the relationship between your answers for LHR₄, RHR₄, and T₄. Explain why this occurs using the shape of the graph.
(e) Estimate the area under the curve from 6 hours to 14 hours with two midpoint rectangles ($M_2$).

(f) Using the units for the area under the curve in part (b) determine how to obtain degrees Fahrenheit. Approximate the average temperature between 6 and 14 hours using part (e).

(g) An air conditioner cooled the house whenever the outside temperature was at or above 80 degrees Fahrenheit. For what values of $t$ was the air conditioner cooling the house?

(h) Estimate the area between the line $y = 80$ and the graph of $F(t)$ for $6 \leq t \leq 18$, using one midpoint between 6 and 18 hours.

(i) The cost of cooling the house accumulates at the rate of $0.05$ per hour for each degree the outside temperature exceeds 80 degrees Fahrenheit. Estimate the total cost using the results from part (h).
Investigating Double Argument Trigonometric/Circular Equations

Consider the equation: \( \cos^2 x = \sin 2x, \; 0 \leq x \leq 2\pi. \)

**Graphical approach:**
1. Graph \( y = \sin 2x, \; 0 \leq x \leq 2\pi \)
2. Graph \( y = 2\sin x \cos x, \; 0 \leq x \leq 2\pi \)
3. Use a graphing calculator to locate the points of intersection between \( y = \cos^2 x \) and \( y = \sin 2x, \; 0 \leq x \leq 2\pi \). Write all points of intersection. The \( x \)-coordinates of the points of intersection are the solutions to the original equation.
4. Write the solutions to the original equation.
5. Graph \( y = 2\sin x \cos x, \; 0 \leq x \leq 2\pi \) and \( y = \sin 2x \).
6. do the graphs compare?
7. Graph \( y = 2\sin x \cos x \) and \( y = \cos^2 x \) and use a graphing calculator to find the points of intersection. How do these points of intersection compare to those found in question?

**Algebraic approach:**
8. \( \cos^2 x = \sin 2x \)
9. Substitute “\( 2\sin x \cos x \)” for “\( \sin 2x \)”.
10. Set the equation equal to zero by subtracting the right-hand member over.
11. Factor \( \cos x \) “out”, set each factor equal to zero, and solve for \( x, \; 0 \leq x \leq 2\pi \).
12. Compare these solutions to #11 with those of #4.

**An incorrect approach:**
13. Graph \( y = \cos x (\cos x - 2\sin x), \; 0 \leq x \leq 2\pi \), and find its zeros. Do these zeros agree with the solutions found in question #11 and #4?
14. Graph \( y = \cos x - 2\sin x, \; 0 \leq x \leq 2\pi \), and find its zeros. Explain how these zeros differ from those found in question #11.
15. Graph \( y = \cos x \), \( 0 \leq x \leq 2\pi \), and find its zeros. Explain why these zeros, combined with those found in question #13, give the solution to the original equation.

16. State why dividing both sides of the equation in #10 by "\( \cos x \)" is not an appropriate method for solving the equation.

An extension:
17. Solve: \( \cos^2 x \geq \sin 2x, 0 \leq x \leq 2\pi \).

Equations and Applications

Solve the following equations.

1. \( 4\sin x \cos x = \sqrt{3}, [0, 2\pi] \)

2. \( \sqrt{2}(\sin x + \cos x) = \sqrt{3}, [0, 2\pi] \)

3. \( \sqrt{1 - \cos 2\theta} = 2\sin^2 \theta, [0^\circ, 360^\circ] \)

4. Find \( \tan x \) if \( \tan 2x = -\frac{3}{4}, \frac{\pi}{2} \leq x \leq \pi \)

5. \( \sin 2x + \cos 3x = 0, [0, 2\pi] \)

6. \( \cos 2x - \sin x = \frac{1}{2}, [0, 2\pi] \)

7. \( \cos 2x - 2\sin^2 x = 0, [0, 2\pi] \)

8. \( \tan (2 \arccos x) = 1, 0 \leq \arccos x \leq \pi \)

9. \( \sin (2\arccos x) = \frac{\sqrt{3}}{2}, 0 \leq \arccos x \leq \frac{\pi}{2} \)

10. \( 4\sin^2 2x \leq 1, [0, \pi] \)
11. Find $x$.

![Diagram of right triangle with sides 8, 12, and hypotenuse labeled $x$.

12. Acceleration due to gravity, $g$, is generally considered to be $9.8 \frac{m}{s^2}$. However, since the earth is ellipsoidal rather than spherical, and since it does not have a smooth surface, changes in latitude and in elevation above sea level alter $g$ according to the following model:

$$g \approx \left[ 9.780327 \left( 1 + 0.0053024 \sin^2 \phi - 0.0000058 \sin^2 2\phi \right) - 3.086 \times 10^{-6} h \right] \frac{m}{s^2},$$

where $\phi$ is the latitude measured in degrees and $h$ is the elevation above sea level measured in meters.

a) Rewrite $g$ in terms of powers of $\sin \phi$ only.

b) Research the values for the latitude and the elevation of your school and then determine $g$ for your location.

c) Find the percent of change between the value calculated in part b above and the standard value of $9.8 \frac{m}{s^2}$.

13. Horizontal distance traveled by a projectile fired from the ground can be modeled by

$$d = \frac{v_0^2 \sin 2\theta}{g},$$

where $v_0$ is the initial velocity, $\theta$ is the launch angle in degrees and $g$ is acceleration due to gravity. (This model neglects air resistance and assumes a flat surface.)

a) A baseball is hit at an initial height of 5 ft with initial velocity of 100 ft/s at 40 degrees to the ground. Will the ball clear a 5 ft fence 310 ft from home plate?

b) What angle will maximize the horizontal distance traveled and why?

c) Will the baseball clear the fence if hit at the angle from part b above?
Related Rates – Triangle Applications

1. A fisherman is standing on the edge of a cliff overlooking a flowing river. The edge of the cliff is 8 feet above the level of the water. The tip of his fishing rod is directly above the edge of the cliff and is 7 feet above the cliff. He lowers his fishing line so that it enters the water at the base of the cliff. The length of his fishing line, \( l \), which is measured from the tip of his fishing rod to the place his line enters the water, and the angle between his line and the water, \( \theta \), are changing as the river carries his line downstream. The distance, \( r \), from the edge of the cliff to the place his line enters the water increases at a rate of \( \frac{1}{2} \text{ ft/sec} \) as the river carries it downstream.

a. Draw a diagram of the situation.

b. Complete the table below where \( A \) represents the area of the triangle formed by the height of the line, the distance from the base of the cliff, and the length of the line.

<table>
<thead>
<tr>
<th>( t ), in sec</th>
<th>( r ), in ft</th>
<th>( A ), in ( \text{ft}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>75</td>
</tr>
</tbody>
</table>

\[ r(t) \]
\[ A(t) \]

c. What is the equation for \( r(t) \)? What is the equation for \( A(t) \)?

d. Sketch the of graph of \( r(t) \) and of \( A(t) \).

e. In the equation \( y = mx + b \), the constant \( m \) is used to represent the slope. In a linear equation, slope may also be written as \( \frac{\Delta y}{\Delta x} \) (average rate of change of \( y \) with respect to \( x \)) or as \( \frac{dy}{dx} \) (instantaneous rate of change of \( y \) with respect to \( x \)). What is the slope of the equation for the distance from the edge of the cliff to the place his line enters the water? Complete the equations shown below using correct units in the answers. How does this rate compare to the one given in the question stem?

\[ m = \frac{\Delta r}{\Delta t} = \frac{dr}{dt} = \]
f. Since the equation for the area of the triangle is linear, use the format shown in part (e) to answer the following: What is the rate of change in the area of the triangle with respect to time?

g. Sketch a graph of the rate of change of the distance the river carries the line downstream with respect to time and a graph of the rate of change of the area with respect to time.

h. Based on this situation, what is an equation using the Pythagorean Theorem?

i. When the length of the fishing line is 17 feet, how far has the river carried the line downstream?

j. The equation \(2r \frac{dr}{dt} = 2l \frac{dl}{dt}\) has been derived using calculus. When the length of the fishing line is 17 feet, what is the rate of change in the length of the fishing line with respect to time?

k. When the river has carried the line 20 feet downstream, what is the length of the fishing line? What is the rate of change in the length of the fishing line with respect to time?

l. Is the length of the fishing line increasing faster when the length is 17 feet or when the length is 25 feet?

m. When the length of the fishing line is 17 feet, what is the value of \(\cos \theta\)?

n. When the length of the fishing line is unknown, what is an equation for \(\tan \theta\) in terms of \(r\)?

o. The equation \(\sec^2 \theta \frac{d\theta}{dt} = \frac{-15}{r^2} \frac{dr}{dt}\), where \(\theta\) is measured in radians, has been derived using calculus. At what rate is the angle between the line and the water changing with respect to time when the length of the fishing line is 17 feet? The answer will be in radians/\(\sec\).
2. A hot air balloon rises vertically at a rate of 10 feet per second from a point on the ground 200 feet from an observer. Let \( h \) be the height of the balloon, let \( \theta \) be the angle of elevation of the balloon, and let \( b \) be the distance from the observer to the balloon.

a. Draw a diagram of the situation.

b. In this situation, \( \frac{dh}{dt} \) is used to represent the rate at which the balloon is rising vertically. What is the value of \( \frac{dh}{dt} \) from the description above? Write your answer as an equation using correct units.

c. What is an equation using the Pythagorean Theorem based on the situation?

d. What is the exact distance from the observer to the balloon when the height of the balloon is 200 feet? When the height of the balloon is \( 200\sqrt{3} \) feet?

e. The equation \( 2h \frac{dh}{dt} = 2b \frac{db}{dt} \) has been derived using calculus. What is the rate of change of the distance of the observer from the balloon at the instant when the balloon is 200 feet above the ground? When the balloon is \( 200\sqrt{3} \) feet above the ground? Re-write the questions using the symbols from the given equation, and then answer the questions.

f. Is the distance between the observer and the balloon changing faster when \( \theta = 45^\circ \) or when \( \theta = \frac{\pi}{3} \) radians?
g. What is the equation for the area of the triangle, \( A \), in terms of the height of the balloon?

h. The equation \( \frac{dA}{dt} = 100 \frac{dh}{dt} \) has been derived using calculus. What is the rate of change of the area of \( \triangle ABC \) with respect to time at the instant when the balloon is 200 feet above the ground? When the balloon is 200\( \sqrt{3} \) feet above the ground? Re-write the questions using the symbols from the given equation, and then answer the questions.

i. What is an equation for the height of the balloon in terms of the angle of elevation?

j. The equation \( \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{200} \frac{dh}{dt} \), where \( \theta \) is measured in radians, has been derived using calculus. What is the rate of change of the angle of elevation of the balloon from the observer with respect to time when the balloon is 200 feet above the ground? When the balloon is 200\( \sqrt{3} \) feet above the ground? Re-write the questions using the symbols from the given equation, and then answer the questions. The answers will be in \( \frac{\text{radians}}{\text{sec}} \).

k. Is the angle increasing faster when \( \theta = 45^\circ \) or when \( \theta = \frac{\pi}{3} \) radians?
3. A patient is considering a procedure where a laser is passed over the skin. For this patient, the laser will start at a perpendicular distance of 50 mm above the skin and will penetrate the skin at a constant depth of 0.5 mm. The angle at which the laser enters the skin will change as the laser moves across the skin’s surface. Let $s$ be the distance the laser has traveled on the skin’s surface, let $b$ be the distance the laser has traveled at the 0.5 mm depth below the skin’s surface, and let $\theta$ be the angle at which the laser enters the skin. The laser will move along the skin at $2 \frac{\text{mm}}{\text{sec}}$.

![Diagram of laser on skin]

a. In this situation, $\frac{ds}{dt}$ is used to represent the rate the laser moves along the skin. What is the value of $\frac{ds}{dt}$ from the description above? Write your answer as an equation using correct units.

b. What is the equation for the distance the laser travels beneath the skin in terms of the distance it travels on the skin?

c. When the laser has traveled 5 mm on the skin, how far has it traveled beneath the skin?

d. How many seconds has the laser been in use if it has traveled 5 mm on the skin?
e. The equation \( \frac{db}{dt} = 1.01 \frac{ds}{dt} \) has been derived using calculus. At what rate is the laser moving beneath the skin?

f. What is the difference, \( x \), between the distance the laser travels on the skin and the distance it travels beneath the skin in terms of \( b \) and \( s \)?

g. The equation \( \frac{dx}{dt} = \frac{db}{dt} - \frac{ds}{dt} \) has been derived using calculus. When the laser has traveled 5 mm on the skin, at what rate is the difference changing?

h. What is the equation for \( \theta \) in terms of the distance the laser has traveled on the skin?

i. When the laser has traveled 5 mm on the skin, what is the measure of \( \theta \) in radians and in degrees?

j. The equation \( \sec^2 \theta \frac{d\theta}{dt} = \frac{-50s}{s^2} \frac{ds}{dt} \), where \( \theta \) is measured in radians, has been derived using calculus. When the laser has traveled 5 mm on the skin, what is the rate of change in the angle at which the laser enters the skin with respect to time? Why is this value negative?
4. Two bicyclists are riding away from a certain point along straight flat routes. The angle between the routes is 120 degrees. Bicyclist A is traveling at 12 mph, and bicyclist B is traveling at 20 mph. Let $a$ represent the distance bicyclist A travels, let $b$ represent the distance bicyclist B travels, and let $p$ represent their distance apart.

a. Draw a diagram of the situation.

b. How far apart are the bicyclists after 15 minutes? After 30 minutes? After 45 minutes?

c. The equation \[ 2p \frac{dp}{dt} = 2a \frac{da}{dt} + 2b \frac{db}{dt} + a \frac{db}{dt} + b \frac{da}{dt} \] has been derived using calculus. After 15 minutes, what is the rate of change in the distance between the bicyclists with respect to time? After 30 minutes? After 45 minutes?

d. Compare the rates at which the distance between the bicyclists is changing for each time period above. What is this comparison? Why does this occur?
What is Best?

1. Best Seat Available

Carlos is planning to attend a concert; however, the only tickets available are single seats on the first aisle of the theater. All of these seats are left of the stage by 20 meters. He knows that these seats are not very good, but he is determined to enjoy the concert. He decides to at least choose the seat with the best viewing angle of the stage. He draws the floor plan for the theater and determines that the width of the stage (CD) is 60 meters. The only seats available are on a line containing the segment AB. Carlos wants $m\angle \beta$ to be as large as possible.

![Diagram of stage and seats]

a) Carlos’ first choice of seats is located on the line containing points A and B at a distance of 20 meters from point B. What is $m\angle \beta$ at this position?

b) Carlos’ second choice is 10 meters from point B. On the sketch draw new triangles BPD and BPC so that point P is 10 meters from point B (much closer than on the current drawing.) What is $m\angle \beta$?

c) Carlos’ third choice is 150 meters from point B. What is $m\angle \beta$ in this position?
d) Explain why the best viewing angle must be between 10 meters and 150 meters.

e) Carlos decides to give up his trial and error search for the best angle. He defines the distance from point P to point B as $x$. He examines the process he used for his first three choices and writes a function to determine $m\angle \beta$. What is the function, $f(x)$?

f) Enter the function as Y1 in a graphing calculator and check the function by recalculating the $m\angle \beta$ from parts (a) and (b).

g) Using a graphing calculator, determine the maximum viewing angle. How far from point B should Carlos sit to obtain this maximum viewing angle?
2. Cheapest Trip

Load the Freight trucking company wants to minimize the cost of driving a truck a distance of 320 miles along a flat, straight highway in west Texas. They have determined that when the truck is traveling at a constant rate of 50 miles per hour, the truck travels 6 miles for each gallon of fuel that it uses. For each increase in speed of 1 mile per hour, the truck’s fuel rate goes down 0.1 mile per gallon. In order to minimize the cost of the trip, the company must determine the optimum constant speed at which the truck should travel.

Three costs are associated with operating the truck:

1. The driver is paid at a rate of $30.50 per hour.
2. The fixed costs of operating the truck are calculated at $28.00 per hour.
3. Fuel costs $2.89 per gallon.

a) What is the fuel cost per hour when the truck travels at 50 mph? Use dimensional analysis in determining the answer. Show your work and include units.

b) If the truck is traveling at 55 miles per hour, at what rate in miles per gallon is it using fuel? What is the fuel cost per hour when the truck travels at 55 miles per hour? Show your work and include units.

c) Include the driver’s hourly salary and the hourly rate for the fixed expense as well as the hourly rate for the fuel to calculate the hourly cost when the truck is traveling at 50 mph and at 55 mph.

d) How much time is required for the truck to travel 320 miles at a constant rate of 50 mph? What is the total cost of the trip when the truck drives at 50 mph? In addition, answer both questions for when the truck is traveling at 55 mph.
e) Determine a function $F(x)$ that can be used to calculate the fuel cost per hour when the truck travels at $x$ miles per hour. Verify that the equation is correct for speeds of 50 mph and 55 mph.

f) Determine a function $C(x)$ that can be used to calculate the total cost per hour of driving the truck at $x$ miles per hour.

g) Determine a function $T(x)$ that can be used to calculate the total cost of driving 320 miles.

h) At what speed should the truck be driven to minimize the cost of the trip?

i) The company manager decides to round the speed to an integer value and then to use the rounded value to calculate the minimum cost of the trip. What is this cost?
3. Shortest Conveyor Belt

The Minimizer Company has been hired to build a conveyor system that can be used to remove debris from a construction site. To control cost, the customer wants the conveyor belt to be as short as possible and to be supported by an existing wall that is 6 feet tall. The debris is to be dropped into a dump truck that will be centered 4 feet past the wall. What is the shortest possible length for the conveyor belt? How far from the base of the wall should the base of the conveyor belt be located?

In the drawing on the right, the conveyor belt begins at point A and ends at point C. The supporting wall that is 6 feet tall is represented by the line segment, DE. The debris is to be dropped above the truck when the truck is centered 4 feet past the base of the wall.

Label AD as $x$, BC as $h$ and AC as $L$ and then develop a function, $L(x)$, to determine the length of the conveyor belt. To determine the minimum length, graph the function with a graphing calculator.

Attempt to determine the solution without working through the questions given below. If you experience difficulty, answer the questions until you can continue without hints.

a) Identify the two right triangles in the drawing and explain why the two triangles are similar.

b) Set up a proportion to relate the lengths of the two vertical sides of the triangles to the lengths of the two horizontal sides and solve the proportion for $h$ in terms of $x$.

c) Write an expression for $L$ in terms of $x$.

d) Graph the function using a reasonable calculator window and determine the point at which the minimum value of the function occurs. What is the shortest possible length for the conveyor belt? How far from the base of the wall should the base of the conveyor belt be located?
4. A Fountain at Minimum Cost
A landscape artist is drawing plans for a fountain to be placed at the entrance to a restaurant known for baking excellent pies. He decides to shape the base of the fountain like a slice of pie. The only restriction that the customer has placed on the design is that the fountain is to have a top surface area of 80 square feet and the fountain pool is to have a height of 2 feet. The curved side of the fountain is to be finished with black river rocks that cost $10.50 per square foot, and the rectangular sides are to be finished with small brown stones that cost $4.50 per square foot. The artist wants to minimize the cost of the finishing stones.

The artist realizes that the surface of the pie is shaped like a sector of a circle, so he will need the following formulas:

(1) The area of the sector of a circle is \( A = \frac{1}{2} r^2 \theta \) where \( r \) is the length of the radius of the circle and \( \theta \) is the central angle measured in radians.

(2) The length of the arc is \( L = r \theta \) where \( \theta \) measured in radians.

He realizes that there are several varying dimensions for his fountain and that he will need to create a cost function that depends on only one of these dimensions, either \( r, \theta \) or \( L \). He decides to use \( r \) as the independent variable for his function.

a) Write an equation for the area of the sector of the circle in terms of both \( r \) and \( L \).

b) Write an equation for the arc length, \( L \), as a function of \( r \).
c) Write the cost function, \( C \), as a function of \( r \). Remember there are two straight rectangular sides and one curved side that must be covered in rocks.

d) Use a graphing calculator to determine the minimum cost of the finishing rocks.

e) Determine all of the measurements for the fountain: \( r \) and \( L \) in feet and \( \theta \) in degrees.
Vectors in Geometry

1. For each of the following problems draw the vectors on a coordinate plane and add the vectors graphically using the head-to-tail method. Calculate the magnitude of the resultant vector and give the direction angle for the resultant vector.

   a) Vector $\vec{u}$ has a magnitude of 3 miles and is directed due east. Vector $\vec{v}$ has a magnitude of 3 miles and is directed due north. What are the magnitude and the direction angle for the resultant vector?

   b) Vector $\vec{u}$ has a magnitude of 500 feet and is directed due south. Vector $\vec{v}$ has a magnitude of 500 feet and is directed due west. What are the magnitude and the direction angle for the resultant vector?

   c) Vector $\vec{u}$ has a magnitude of $5\sqrt{3}$ feet and is directed due west. Vector $\vec{v}$ has a magnitude of 5 feet and is directed due north. What are the magnitude and the direction angle for the resultant vector?

   d) Vector $\vec{u}$ has a magnitude of 300 miles and is directed due east. Vector $\vec{v}$ has a magnitude of $300\sqrt{3}$ miles and is directed due north. What are the magnitude and the direction angle for the resultant vector?

   e) Vector $\vec{u}$ has a magnitude of 9 cm and is directed to the right. Vector $\vec{v}$ has a magnitude of $9\sqrt{3}$ cm and is directed downward. What are the magnitude and the direction angle for the resultant vector?

   f) Vector $\vec{u}$ has a magnitude of 15 meters and is directed to the left. Vector $\vec{v}$ has a magnitude of 3 meters and is directed to the right. What are the magnitude and the direction angle for the resultant vector?

   g) Vector $\vec{u}$ has a magnitude of 3 cm and is directed upward. Vector $\vec{v}$ has a magnitude of 11 cm and is directed downward. What are the magnitude and the direction angle for the resultant vector?

   h) Vector $\vec{u}$ has a magnitude of 10 inches with a direction angle of 45°. Vector $\vec{v}$ has a magnitude of 10 inches and a direction angle of 225°. What are the magnitude and the direction angle for the resultant vector?
i) Vector \( \mathbf{u} \) has a magnitude of \( 10\sqrt{2} \) meters and is directed to the southwest. Vector \( \mathbf{v} \) has a magnitude of 10 meters and is directed eastward. What are the magnitude and the direction angle for the resultant vector?

j) Vector \( \mathbf{u} \) has a magnitude of 24 miles with a direction angle of 150°. Vector \( \mathbf{v} \) has a magnitude of \( 12\sqrt{3} \) miles with a direction angle of 0°. What are the magnitude and the direction for the resultant vector?

2. Let \( \mathbf{c} \) be the resultant of the two vectors \( \mathbf{a} \) and \( \mathbf{b} \). Vector \( \mathbf{b} \) has a magnitude of \( b \) units directed to the east, and vector \( \mathbf{a} \) has as a magnitude of \( a \) units directed to the west. What are the three possibilities for the resultant’s direction angle and magnitude?

3. The vectors given below represent forces acting on a point at the origin of a coordinate plane. Determine the direction and magnitude of the resultant vector.
   a) Vector \( \mathbf{a} \) has a magnitude of 10 lbs and a direction angle of 60°, and vector \( \mathbf{b} \) has a magnitude of 10 lbs and a direction angle of 300°.
   b) Vector \( \mathbf{a} \) has a magnitude of 24 kg and a direction angle of 0°, and vector \( \mathbf{b} \) has a magnitude of 12 kg and a direction angle of 120°.
   c) Vector \( \mathbf{a} \) has a magnitude of \( 10\sqrt{2} \) lbs and a direction angle of 45°, and vector \( \mathbf{b} \) has a magnitude of \( 10\sqrt{2} \) lbs and a direction angle of 135°

4. Given that \( O \) is a point at the origin on which two forces are acting, what force must be combined with a force of 1 lb at a direction angle of 0° in order for the resultant force to have a magnitude of \( \sqrt{3} \) lb. and a direction angle of 90°?

5. Given that \( O \) is a point at the origin on which two forces are acting, what force must be combined with a force of 20 kg at a direction angle of 60° in order for the resultant force to have a magnitude of \( 20\sqrt{3} \) kg and a direction angle of 90°?

6. Determine two forces with direction angles of 90° and 180°, respectively, that result in a vector \( b \) units in magnitude with a direction angle of 120°.
Applications of Vectors

For the following, bearing will be measured as degrees clockwise from the north.
For 1 and 2 find the resultant of the two given displacements. Express the answer as a distance from the starting point and a bearing (clockwise from the north) from the starting point to the end point.

1. 11 miles north \((0^\circ)\) followed by 6 miles along a bearing of \(70^\circ\).

2. 8 kilometers east \((90^\circ)\) followed by 6 kilometers along a bearing of \(210^\circ\).

3. A ship sails 50 miles on a bearing of \(20^\circ\), then 30 miles further on a bearing of \(80^\circ\). Find the resultant displacement vector as a distance and the bearing of this vector.

4. A plane flies 30 miles on a bearing of \(290^\circ\), then turns and flies 40 miles on a bearing of \(50^\circ\). Find the displacement vector, determine its magnitude and the bearing.

5. An expedition walks 4 kilometers from camp on bearing of \(30^\circ\), then turns and walks 10 km on a bearing of \(160^\circ\). Find the magnitude and bearing of the new location relative to the beginning location.
Because of wind, a plane’s ground speed (actual speed relative to the ground) might differ from its air speed (speed in still air). In addition, a plane’s true course (the direction in which it actually travels), might differ from its heading (the direction in which it is pointed).

6. Find the ground speed of an airplane with air speed 480 km/hr and heading $90^\circ$ if a wind blowing south has a velocity of 40 km/hr.

7. A plane’s heading is $160^\circ$ and its air speed is 350 mph. If a wind is blowing east at 20 mph, what are the plane’s ground speed and true course?

8. A plane with a heading of $50^\circ$ has an air speed of 400 mph. If a 35 mph wind is blowing from the north, what are the plane’s ground speed and true course?

9. An ocean liner has a heading of $250^\circ$ and a speed of 12 knots. If the ship’s true course is $235^\circ$ and its speed relative to land is 10 knots, what are the speed and direction of the water current? (Hint: you are given the resultant vector)

10. An airplane must fly at a ground speed of 450 km/hr on course $70^\circ$ to be on schedule. The wind velocity is 25 km/hr in a bearing of $40^\circ$. Find the necessary heading and the air speed to maintain the course.
Parametric Equations

1. Anytown High School is planning a play. The script calls for two characters to meet on stage. Lauren starts at the point (0 feet, 6 feet) and travels horizontally at a rate of 1 foot per second. Alex starts at the point (4 feet, 0 feet) and travels vertically at a rate of 2 feet per second. If Alex and Lauren start walking at the same time, will they meet?

   a) Use the grid to graph each walk.

   b) From the graph in part (a), can you determine if Alex and Lauren meet? Explain your answer.

   c) Complete the table of values for Lauren and Alex.

<table>
<thead>
<tr>
<th>Lauren</th>
<th>Alex</th>
</tr>
</thead>
<tbody>
<tr>
<td>time (in seconds)</td>
<td>x (horizontal)</td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
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<td>6</td>
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<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>t</td>
<td></td>
</tr>
</tbody>
</table>

   d) Can you tell from the table if Lauren and Alex meet? Explain your answer.
e) Write a pair of equations for Lauren’s horizontal and vertical position in terms of the third variable, or parameter, time. These are called parametric equations.

f) Write a pair of equations for Alex’s horizontal and vertical position in terms of the third variable, or parameter, time.

The three variables in a pair of parametric equations can represent anything, but in this problem x and y represent positions and t represents time.

Using the Calculator

<table>
<thead>
<tr>
<th>MODE</th>
<th>Y =</th>
<th>WINDOW</th>
<th>GRAPH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal Sci Eng</td>
<td>Plot1, Plot2, Plot3</td>
<td>WINDOW</td>
<td>GRAPH</td>
</tr>
<tr>
<td>Float 0123456789</td>
<td>X1T, X1T</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Radian Degree</td>
<td>Y1=2X</td>
<td>Tmin=8</td>
<td>Tstep=.1</td>
</tr>
<tr>
<td>Func Par Pol Seq</td>
<td>Y2=2T</td>
<td>Xmin=0</td>
<td>Xmax=6</td>
</tr>
<tr>
<td>Connected Dot</td>
<td>Y3=t</td>
<td>Xscl=1</td>
<td>Ymin=0</td>
</tr>
<tr>
<td>Sequential Simul</td>
<td>Y4=</td>
<td>Xmax=1</td>
<td>Ymax=10</td>
</tr>
<tr>
<td>Real a+bi re^at</td>
<td></td>
<td>Xscl=1</td>
<td>Yscl=1</td>
</tr>
<tr>
<td>Full Horiz G-T</td>
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</tbody>
</table>

g) The script is incorrect since Alex and Lauren do not meet on stage at the same time. The director of the play decides that Lauren and Alex should meet after 4 seconds. Write a new pair of parametric equations that will produce this result.
2. A bug is crawling up a wall. He starts at the bottom left corner of the wall. He is crawling along the wall 2 inches to the right every second and 1 inch up every second. A lizard is 6 inches above the floor on the left corner of the wall. He is traveling .5 inches down every second and 2 inches to the right every second.

a) The graph to the right shows the paths of the lizard and the bug.

From the graph, can you determine if the lizard and the bug meet?

Explain your answer.

b) Complete the table of values for the bug and the lizard.

<table>
<thead>
<tr>
<th>Bug</th>
<th>Lizard</th>
</tr>
</thead>
<tbody>
<tr>
<td>time (in seconds)</td>
<td>x (horizontal)</td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
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<td>7</td>
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<tr>
<td>t</td>
<td></td>
</tr>
</tbody>
</table>

c) From the table of values in part (b), can you determine if the bug and the lizard meet?

d) When do they meet? What is their position when they meet?

e) What variable can you see in the table of values in part (b) that is hidden in the graph in part (a)?
f) Write a pair of equations for the bug’s position in terms of the third variable, or parameter, *time*.

g) Write a pair of equations for the lizard’s position in terms of the third variable, or parameter, *time*.

h) Use your calculator to graph the parametric equations that you wrote in parts (f) and (g).

Complete the table of values below.

<table>
<thead>
<tr>
<th>Bug</th>
<th>Lizard</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tr>
</tbody>
</table>

i) Write a position function for the vertical distance *y* in terms of the horizontal distance *x* for the position of the bug.

j) Write a position function for the vertical distance *y* in terms of the horizontal distance *x* for the position of the lizard.

k) Write new parametric equations so that the bug will travel the same path but he will pass the point where his path crosses the lizard’s path after the lizard. Use your calculator to check your answer.

l) Will the parametric equations you wrote in part (k) change the position function that you wrote in part (i)? Explain your answer.
3. Jeremy begins walking across a field. He is traveling northeast at a rate of 0.5 feet per second east and 0.25 feet per second north. Denise is 20 feet north and 4 feet east of Jeremy’s original position. She is traveling southeast at a rate of 0.4 feet per second east and 0.8 feet per second south. Let Jeremy’s beginning position be at the origin.

a) Write a pair of parametric equations that represent Jeremy’s walk and Denise’s walk.

b) Use your calculator to help you determine if Jeremy and Denise meet.

c) Write a position function for the vertical distance $y$ in terms of the horizontal distance $x$ for the position of Jeremy.

d) Write a position function for the vertical distance $y$ in terms of the horizontal distance $x$ for the position of Denise.

e) Graph the functions that you wrote in parts (c) and (d) on a coordinate plane.

f) Use the position functions that you wrote in parts (c) and (d) to determine where Jeremy and Denise cross paths.

g) Find the time that each person reaches the point where their paths cross.

h) Use the information you found in parts (f) and (g) to support your observation in part b.
Motion Defined Parametrically

1. A particle moves so that its position measured in feet at any time \( t \) seconds, \( 0 \leq t \leq 4 \), is given by
   \[
   x(t) = 2t - 1
   \]
   \[
   y(t) = t^2 - 1.
   \]
   a) Complete the table below.
   
<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
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<td>4</td>
<td></td>
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</tbody>
</table>

   b) Graph the path of the particle for \( 0 \leq t \leq 4 \) on the calculator then transfer the graph to paper, making sure that the scale is correct.

   c) On your graph, plot the points from the table and connect the consecutive points with line segments.

   d) Calculate the length of the line segments drawn between the consecutive points.

   e) The distance traveled by the particle that moves on the path can be approximated by summing the distances along the line segments between the consecutive points. Approximate the distance traveled by the particle.

   f) Do you think that your approximation is too large or too small? Explain your reasoning.

   g) The speed of the particle can be calculated as distance divided by time. What is the particle’s approximate average speed over the time interval \([0,4]\)? Be sure to state the units.
h) Write an equation for \( y \) in terms of \( x \).

i) Find the domain and range for the function \( y = f(x) \) that you found in \((h)\). (Remember that the domain and range are determined so that the graph will be the same as it is in parametric form.)

2. A particle moves so that its position at any time \( t \) seconds such that \( 0 \leq t \leq 4 \) is given by
   \[ x(t) = t^2 - 1 \]
   \[ y(t) = 2t - 1. \]

a) Complete the table below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
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<tr>
<td>4</td>
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</tbody>
</table>

b) Using the calculator, graph the path of the particle for \( 0 \leq t \leq 4 \) then transfer the graph to paper.

c) Explain the relationship between this new function and the function in question 1.

d) Approximate the distance traveled from \( t = 0 \) to \( t = 4 \). Explain why no calculations are necessary.

e) Will the approximate average speed be the same as that in question 1g?
f) Write an equation for $y$ in terms of $x$. 


g) State the domain and range for the new function.

3. A particle moves in the plane so that its position is given by $x(t) = t$ and $y(t) = (t^2 - 1)(t + b)$ for $0 \leq t \leq 5$. The position of the particle at $t = 0$ is (0,5). Assume that $x$ and $y$ are measured in centimeters and that $t$ is measured in seconds.

a) Find $b$.

b) Graph the path of the particle.

c) The equation for $x$ in terms of $t$ is $x(t) = t$. What is the rate of change of $x$ with respect to $t$? Be sure to state the units.

d) This particle is always moving to the right. How is this confirmed by the answer in 3c?

e) Approximate the total distance traveled using $\Delta t = 1$ beginning at $t = 0$ and ending at $t = 5$.

f) Approximate the average speed of the particle over $0 \leq t \leq 5$.

g) Write an equation for $y$ in terms of $x$.

h) Based on the work done in this activity, explain how to approximate the distance that a particle will travel along a curve, and also explain how to approximate the average speed of a particle over a specified time interval.
Graphing Polar Equations

On problems 1 through 4, put your graphing calculator in POLAR mode and RADIAN mode with a window of $0 \leq \theta \leq 2\pi$.

1. Graph the following equations on your calculator and sketch the graphs. What do you notice about these graphs?
   
   \[ r = 2\cos\theta \]
   
   \[ r = -3\cos\theta \]
   
   \[ r = 2\sin\theta \]
   
   \[ r = -3\sin\theta \]

2. Graph the following equations on your calculator, sketch the graphs and answer the following:
   
   \[ r = 2 + 2\cos\theta \]
   
   \[ r = 1 + 2\cos\theta \]
   
   \[ r = 2 + \cos\theta \]
   
   \[ r = 2 + 2\sin\theta \]
   
   \[ r = 1 + 2\sin\theta \]
   
   \[ r = 2 + \sin\theta \]

   a) Which graphs go through the pole (origin)? Why?
   
   b) Which ones do not go through the pole? Why?
   
   c) Which ones have an inner loop?
   
   d) What causes the inner loops to happen?

   (Hint: Go to FORMAT and set your calculator to show the polar graphing coordinates when you trace.)
3. Graph the following equations on your calculator.

\[ r = 3 \cos 3\theta \]
\[ r = 2 \cos 5\theta \]
\[ r = 3 \sin 3\theta \]
\[ r = 2 \sin 5\theta \]

What do you notice about these graphs?

4. Graph the following equations on your calculator.

\[ r = 2 \cos 2\theta \]
\[ r = 3 \cos 4\theta \]
\[ r = 2 \sin 2\theta \]
\[ r = 3 \sin 4\theta \]

What do you notice about these graphs?

Do not use your graphing calculator for problems 5 - 12. Convert the following equations to rectangular form, and graph.

5. \( r \sin \theta = -2 \)

6. \( r = 3 \sec \theta \)

7. \( r = 2 \)

8. Make a table and graph \( r = 2 \sin 3\theta \). Name the points in polar coordinates where each petal begins and ends.

9. Make a table and graph \( r = 1 + 2 \sin \theta \). Name the points in polar coordinates where the inner loop begins and ends.

Make a table and graph each curve, and find the points of intersection in polar coordinates.

10. \( r = 4 \sin \theta \) and \( r = 2 \)

11. \( r = 2 \cos \theta \) and \( r = \cos \theta \)

12. \( r = 2(1 - \sin \theta) \) and \( r = 2 \)
Special Points on Polar Curves and Intersections of Two Polar Curves

Example 1:
Graph the limaçon \( r = 1 + 2 \cos \theta \), and find the points at which the inner loop begins and ends.

Set \( r \) equal to zero and solve for \( \theta \).

\[
1 + 2 \cos \theta = 0
\]

\[
\cos \theta = -\frac{1}{2}
\]

\[
\theta = \frac{2\pi}{3}, \frac{4\pi}{3}
\]

The inner loop begins at the point \( \left( 0, \frac{2\pi}{3} \right) \) and ends at the point \( \left( 0, \frac{4\pi}{3} \right) \).

Example 2:
Graph the rose \( r = 2 \sin 3\theta \), and find the points at which the petals begin and end.

Set \( r \) equal to zero and solve for \( \theta \).

\[
2 \sin 3\theta = 0
\]

\[
3\theta = \pi n
\]

\[
\theta = \frac{\pi n}{3}
\]

\[
\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi
\]

The petals begin and end at \( (0, 0), \left( 0, \frac{\pi}{3} \right), \left( 0, \frac{2\pi}{3} \right), (0, \pi) \).

Note: The end point of one petal is the beginning point of a different petal.
Example 3:
Graph \( r = 3 \sin \theta \) and \( r = 2 - \sin \theta \), and find the points of intersection.

Set the equations equal to each other, and solve for \( \theta \).

\[
3 \sin \theta = 2 - \sin \theta
\]

\[
4 \sin \theta = 2
\]

\[
\sin \theta = \frac{1}{2}
\]

\[
\theta = \frac{\pi}{6}, \frac{5\pi}{6}
\]

If \( \theta = \frac{\pi}{6}, r = \frac{3}{2} \). If \( \theta = \frac{5\pi}{6}, r = \frac{3}{2} \).

The points of intersection are \( \left( \frac{3}{2}, \frac{\pi}{6} \right) \) and \( \left( \frac{3}{2}, \frac{5\pi}{6} \right) \).
Activity:

Graph the following limaçons and determine the points at which the inner loop begins and ends.

1. \( r = 1 + 2 \sin \theta \)
2. \( r = \frac{1}{2} - \cos \theta \)
3. \( r = 2 + 3 \cos \theta \)

Graph the following roses, and find the points at which the petals begin and end.

4. \( r = \cos 3\theta \)
5. \( r = 3 \sin 2\theta \)
6. \( r = 2 \sin 4\theta \)

Graph the following polar curves, and find their points of intersection.

7. \( r = 2 + 2 \cos \theta, \; r = 3 \)
8. \( r = 1 - \sin \theta, \; r = 1 \)
9. \( r = 4 \sin \theta, \; r = 2 \)
10. \( r = 2 \sin \theta, \; r = 2 \cos \theta \)
11. \( r = \cos \theta - 1, \; r = 3 \cos \theta \)
12. \( r = 1 + \cos \theta, \; r = 1 - \cos \theta \)
13. \( r = 3 - 2 \sin \theta, \; r = 3 - 2 \cos \theta \)
14. \( r = 4, \; r = 2 \sec \theta \)
15. \( r = 4 \cos 2\theta, \; r = 4 \sin 2\theta \)
16. \( r = 6 \sin 2\theta, \; r = 3 \)