CHAPTER 13 Functions of Several Variables

Section 13.1

Introduction to Functions of Several Variables

- Understand the notation for a function of several variables.
- Sketch the graph of a function of two variables.
- Sketch level curves for a function of two variables.
- Sketch level surfaces for a function of three variables.
- Use computer graphics to graph a function of two variables.

Functions of Several Variables

So far in this text, you have dealt only with functions of a single (independent) variable. Many familiar quantities, however, are functions of two or more variables. For instance, the work done by a force and the volume of a right circular cylinder \((V = \pi r^2h)\) are both functions of two variables. The volume of a rectangular solid \((V = lwh)\) is a function of three variables. The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

**Function of two variables**

\[ z = f(x, y) = x^2 + xy \]

2 variables

**Function of three variables**

\[ w = f(x, y, z) = x + 2y - 3z \]

3 variables

**Definition of a Function of Two Variables**

Let \(D\) be a set of ordered pairs of real numbers. If to each ordered pair \((x, y)\) in \(D\) there corresponds a unique real number \(f(x, y)\), then \(f\) is called a function of \(x\) and \(y\). The set \(D\) is the domain of \(f\), and the corresponding set of values for \(f(x, y)\) is the range of \(f\).

For the function given by \(z = f(x, y)\), \(x\) and \(y\) are called the independent variables and \(z\) is called the dependent variable.

Somerville was interested in the problem of creating geometric models for functions of several variables. Her most well-known book, *The Mechanics of the Heavens*, was published in 1831.
**EXAMPLE 1  Domains of Functions of Several Variables**

Find the domain of each function.

a. \( f(x, y) = \sqrt{x^2 + y^2 - 9} \)

b. \( g(x, y, z) = \frac{x}{\sqrt{9 - x^2 - y^2 - z^2}} \)

**Solution**

a. The function \( f \) is defined for all points \((x, y)\) such that \( x \neq 0 \) and \( x^2 + y^2 \geq 9 \).

So, the domain is the set of all points lying on or outside the circle \( x^2 + y^2 = 9 \), except those points on the \( y \)-axis, as shown in Figure 13.1.

b. The function \( g \) is defined for all points \((x, y, z)\) such that \( x^2 + y^2 + z^2 < 9 \).

Consequently, the domain is the set of all points \((x, y, z)\) lying inside a sphere of radius 3 that is centered at the origin.

**Try It**

**Exploration A**

Functions of several variables can be combined in the same ways as functions of single variables. For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

\[
(f \pm g)(x, y) = f(x, y) \pm g(x, y) \quad \text{Sum or difference}
\]

\[
(fg)(x, y) = f(x, y)g(x, y) \quad \text{Product}
\]

\[
\frac{f}{g}(x, y) = \frac{f(x, y)}{g(x, y)} \quad \text{Quotient}
\]

You cannot form the composite of two functions of several variables. However, if \( h \) is a function of several variables and \( g \) is a function of a single variable, you can form the **composite** function \((g \circ h)(x, y)\) as follows.

\[
(g \circ h)(x, y) = g(h(x, y)) \quad \text{Composition}
\]

The domain of this composite function consists of all \((x, y)\) in the domain of \( h \) such that \( h(x, y) \) is in the domain of \( g \). For example, the function given by

\[
f(x, y) = \sqrt{16 - 4x^2 - y^2}
\]

can be viewed as the composite of the function of two variables given by \( h(x, y) = 16 - 4x^2 - y^2 \) and the function of a single variable given by \( g(u) = \sqrt{u} \). The domain of this function is the set of all points lying on or inside the ellipse given by \( 4x^2 + y^2 = 16 \).

A function that can be written as a sum of functions of the form \( cx^m y^n \) (where \( c \) is a real number and \( m \) and \( n \) are nonnegative integers) is called a **polynomial function** of two variables. For instance, the functions given by

\[
f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2
\]

are polynomial functions of two variables. A **rational function** is the quotient of two polynomial functions. Similar terminology is used for functions of more than two variables.
The Graph of a Function of Two Variables

As with functions of a single variable, you can learn a lot about the behavior of a function of two variables by sketching its graph. The graph of a function of two variables is the set of all points \((x, y, z)\) for which \(z = f(x, y)\) and \((x, y)\) is in the domain of \(f\). This graph can be interpreted geometrically as a surface in space, as discussed in Sections 11.5 and 11.6. In Figure 13.2, note that the graph of \(z = f(x, y)\) is a surface whose projection onto the \(xy\)-plane is \(D\), the domain of \(f\). To each point \((x, y)\) in \(D\) there corresponds a point \((x, y, z)\) on the surface, and, conversely, to each point \((x, y, z)\) on the surface there corresponds a point \((x, y)\) in \(D\).

**EXAMPLE 2** Describing the Graph of a Function of Two Variables

What is the range of \(f(x, y) = \sqrt{16 - 4x^2 - y^2}\)? Describe the graph of \(f\).

**Solution** The domain \(D\) implied by the equation for \(f\) is the set of all points \((x, y)\) such that \(16 - 4x^2 - y^2 \geq 0\). So, \(D\) is the set of all points lying on or inside the ellipse given by

\[
\frac{x^2}{4} + \frac{y^2}{16} = 1. \quad \text{Ellipse in the xy-plane}
\]

The range of \(f\) is all values \(z = f(x, y)\) such that \(0 \leq z \leq \sqrt{16}\) or \(0 \leq z \leq 4\). \(\text{Range of } f\)

A point \((x, y, z)\) is on the graph of \(f\) if and only if

\[
z = \sqrt{16 - 4x^2 - y^2} = 4x^2 + y^2 + z^2 = 16
\]

\[
\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1, \quad 0 \leq z \leq 4.
\]

From Section 11.6, you know that the graph of \(f\) is the upper half of an ellipsoid, as shown in Figure 13.3.

**Try It**

To sketch a surface in space *by hand*, it helps to use traces in planes parallel to the coordinate planes, as shown in Figure 13.3. For example, to find the trace of the surface in the plane \(z = 2\), substitute \(z = 2\) in the equation \(z = \sqrt{16 - 4x^2 - y^2}\) and obtain

\[
2 = \sqrt{16 - 4x^2 - y^2} \implies \frac{x^2}{3} + \frac{y^2}{12} = 1.
\]

So, the trace is an ellipse centered at the point \((0, 0, 2)\) with major and minor axes of lengths \(4\sqrt{3}\) and \(2\sqrt{3}\).

Traces are also used with most three-dimensional graphing utilities. For instance, Figure 13.4 shows a computer-generated version of the surface given in Example 2. For this graph, the computer took 25 traces parallel to the \(xy\)-plane and 12 traces in vertical planes.

If you have access to a three-dimensional graphing utility, use it to graph several surfaces.
**Level Curves**

A second way to visualize a function of two variables is to use a scalar field in which the scalar \( z = f(x, y) \) is assigned to the point \( (x, y) \). A scalar field can be characterized by level curves (or contour lines) along which the value of \( f(x, y) \) is constant. For instance, the weather map in Figure 13.5 shows level curves of equal pressure called **isobars**. In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms**, as shown in Figure 13.6. Another common use of level curves is in representing electric potential fields. In this type of map, the level curves are called **equipotential lines**.

Contour maps are commonly used to show regions on Earth’s surface, with the level curves representing the height above sea level. This type of map is called a **topographic map**. For example, the mountain shown in Figure 13.7 is represented by the topographic map in Figure 13.8. View the animation to see this more clearly.

A contour map depicts the variation of \( z \) with respect to \( x \) and \( y \) by the spacing between level curves. Much space between level curves indicates that \( z \) is changing slowly, whereas little space indicates a rapid change in \( z \). Furthermore, to give a good three-dimensional illusion in a contour map, it is important to choose \( c \)-values that are evenly spaced.
**EXAMPLE 3** Sketching a Contour Map

The hemisphere given by \( f(x, y) = \sqrt{64 - x^2 - y^2} \) is shown in Figure 13.9. Sketch a contour map for this surface using level curves corresponding to \( c = 0, 1, 2, \ldots, 8 \).

**Solution** For each value of \( c \), the equation given by \( f(x, y) = c \) is a circle (or point) in the \( xy \)-plane. For example, when \( c = 0 \), the level curve is \( x^2 + y^2 = 64 \) Circle of radius 8 which is a circle of radius 8. Figure 13.10 shows the nine level curves for the hemisphere.

![Hemisphere](Figure 13.9)

**EXAMPLE 4** Sketching a Contour Map

The hyperbolic paraboloid given by \( z = y^2 - x^2 \)

is shown in Figure 13.11. Sketch a contour map for this surface.

**Solution** For each value of \( c \), let \( f(x, y) = c \) and sketch the resulting level curve in the \( xy \)-plane. For this function, each of the level curves \( (c \neq 0) \) is a hyperbola whose asymptotes are the lines \( y = \pm x \). If \( c < 0 \), the transverse axis is horizontal. For instance, the level curve for \( c = -4 \) is given by

\[
\frac{x^2}{c^2} - \frac{y^2}{c^2} = 1. \quad \text{Hyperbola with horizontal transverse axis}
\]

If \( c > 0 \), the transverse axis is vertical. For instance, the level curve for \( c = 4 \) is given by

\[
\frac{y^2}{c^2} - \frac{x^2}{c^2} = 1. \quad \text{Hyperbola with vertical transverse axis}
\]

If \( c = 0 \), the level curve is the degenerate conic representing the intersecting asymptotes, as shown in Figure 13.12.

![Hyperbolic paraboloid](Figure 13.11)

**Try It**

**Open Exploration**
One example of a function of two variables used in economics is the **Cobb-Douglas production function**. This function is used as a model to represent the number of units produced by varying amounts of labor and capital. If \( x \) measures the units of labor and \( y \) measures the units of capital, the number of units produced is given by

\[
f(x, y) = Cx^a y^{1-a}
\]

where \( C \) and \( a \) are constants with \( 0 < a < 1 \).

**EXAMPLE 5  The Cobb-Douglas Production Function**

A toy manufacturer estimates a production function to be \( f(x, y) = 100x^{0.6}y^{0.4} \), where \( x \) is the number of units of labor and \( y \) is the number of units of capital. Compare the production level when \( x = 1000 \) and \( y = 500 \) with the production level when \( x = 2000 \) and \( y = 1000 \).

**Solution**  When \( x = 1000 \) and \( y = 500 \), the production level is

\[
f(1000, 500) = 100(1000^{0.6})(500^{0.4}) \approx 75,786.
\]

When \( x = 2000 \) and \( y = 1000 \), the production level is

\[
f(2000, 1000) = 100(2000^{0.6})(1000^{0.4}) = 151,572.
\]

The level curves of \( z = f(x, y) \) are shown in Figure 13.13. Note that by doubling both \( x \) and \( y \), you double the production level (see Exercise 79).

**Level Surfaces**

The concept of a level curve can be extended by one dimension to define a **level surface**. If \( f \) is a function of three variables and \( c \) is a constant, the graph of the equation \( f(x, y, z) = c \) is a **level surface** of the function \( f \), as shown in Figure 13.14.

With computers, engineers and scientists have developed other ways to view functions of three variables. For instance, Figure 13.15 shows a computer simulation that uses color to represent the pressure waves of a high-speed train traveling through a tunnel.
EXAMPLE 6  **Level Surfaces**

Describe the level surfaces of the function

\[ f(x, y, z) = 4x^2 + y^2 + z^2. \]

**Solution**  Each level surface has an equation of the form

\[ 4x^2 + y^2 + z^2 = c. \]

Equation of level surface

So, the level surfaces are ellipsoids (whose cross sections parallel to the
yz-plane are circles). As \( c \) increases, the radii of the circular cross sections increase
according to the square root of \( c \). For example, the level surfaces corresponding to the
values \( c = 0 \), \( c = 4 \), and \( c = 16 \) are as follows.

\[
\begin{align*}
4x^2 + y^2 + z^2 &= 0 & \text{Level surface for } c = 0 \text{ (single point)} \\
\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} &= 1 & \text{Level surface for } c = 4 \text{ (ellipsoid)} \\
\frac{x^2}{16} + \frac{y^2}{16} + \frac{z^2}{16} &= 1 & \text{Level surface for } c = 16 \text{ (ellipsoid)}
\end{align*}
\]

These level surfaces are shown in Figure 13.16.

NOTE  If the function in Example 6 represented the temperature at the point \( (x, y, z) \), the level surfaces shown in Figure 13.16 would be called isothermal surfaces.

**Computer Graphics**

The problem of sketching the graph of a surface in space can be simplified by using a computer. Although there are several types of three-dimensional graphing utilities, most use some form of trace analysis to give the illusion of three dimensions. To use such a graphing utility, you usually need to enter the equation of the surface, the region in the xy-plane over which the surface is to be plotted, and the number of traces to be taken. For instance, to graph the surface given by

\[ f(x, y) = (x^2 + y^2)e^{1-x^2-y^2} \]

you might choose the following bounds for \( x \), \( y \), and \( z \).

- \( -3 \leq x \leq 3 \)  \hspace{1cm} \text{Bounds for } x
- \( -3 \leq y \leq 3 \)  \hspace{1cm} \text{Bounds for } y
- \( 0 \leq z \leq 3 \)  \hspace{1cm} \text{Bounds for } z

Figure 13.17 shows a computer-generated graph of this surface using 26 traces taken parallel to the \( yz \)-plane. To heighten the three-dimensional effect, the program uses a “hidden line” routine. That is, it begins by plotting the traces in the foreground (those corresponding to the largest \( x \)-values), and then, as each new trace is plotted, the program determines whether all or only part of the next trace should be shown.

The graphs on page 891 show a variety of surfaces that were plotted by computer. If you have access to a computer drawing program, use it to reproduce these surfaces. Remember also that the three-dimensional graphics in this text can be viewed and rotated.
Three different views of the graph of \( f(x, y) = (2 - y^2 + x^2)e^{1 - x^2 - (y^2/4)} \)

Traces and level curves of the graph of \( f(x, y) = \frac{-4x}{x^2 + y^2 + 1} \)

Three functions are shown:
- \( f(x, y) = \sin x \sin y \)
- \( f(x, y) = -\frac{1}{\sqrt{x^2 + y^2}} \)
- \( f(x, y) = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} \)
Exercises for Section 13.1

The symbol \( \text{\textcircled{H}} \) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on [S] to view the complete solution of the exercise.

Click on [M] to print an enlarged copy of the graph.

In Exercises 1–4, determine whether \( z \) is a function of \( x \) and \( y \).

1. \( x^2 + y^2 - z = 10 \)  
2. \( x^2 + 2xy - y^2 = 4 \)  
3. \( x^2 + y^2 + z^2 = 1 \)  
4. \( z + x \ln y - 8 = 0 \)

In Exercises 5–16, find and simplify the function values.

5. \( f(x, y) = xy \)  
(a) \( (3, 2) \)  
(b) \( (-1, 4) \)  
(c) \( (30, 5) \)  
(d) \( (5, y) \)  
(e) \( (x, 2) \)  
(f) \( (t, t) \)  
6. \( f(x, y) = 4 - x^2 - 4y^2 \)  
(a) \( (0, 0) \)  
(b) \( (0, 1) \)  
(c) \( (2, 3) \)  
(d) \( (1, y) \)  
(e) \( (x, 0) \)  
(f) \( (t, t) \)  
7. \( f(x, y) = xe^y \)  
(a) \( (5, 0) \)  
(b) \( (3, 2) \)  
(c) \( (2, -1) \)  
(d) \( (5, y) \)  
(e) \( (x, 2) \)  
(f) \( (t, t) \)  
8. \( g(x, y) = \ln|\ln x + y| \)  
(a) \( (2, 3) \)  
(b) \( (5, 6) \)  
(c) \( (e, 0) \)  
(d) \( (0, 1) \)  
(e) \( (2, -3) \)  
(f) \( (e, e) \)  
9. \( h(x, y, z) = \frac{xy}{z} \)  
(a) \( (2, 3, 9) \)  
(b) \( (1, 0, 1) \)  
(c) \( (-2, 3, 4) \)  
(d) \( (5, 4, -6) \)  
10. \( f(x, y, z) = \sqrt{x + y + z} \)  
(a) \( (0, 5, 4) \)  
(b) \( (6, 8, -3) \)  
(c) \( (4, 6, 2) \)  
(d) \( (10, -4, -3) \)  
11. \( f(x, y) = x \sin y \)  
(a) \( \left(\frac{2}{3}, \frac{\pi}{4}\right) \)  
(b) \( (3, 1) \)  
(c) \( (-3, \frac{\pi}{3}) \)  
(d) \( (4, \frac{\pi}{2}) \)  
12. \( V(r, h) = \pi r^2 h \)  
(a) \( (3, 10) \)  
(b) \( (5, 2) \)  
(c) \( (4, 8) \)  
(d) \( (6, 4) \)  
13. \( g(x, y) = \int_0^1 (2t - 3) \, dt \)  
(a) \( (0, 4) \)  
(b) \( (1, 4) \)  
(c) \( \left(\frac{3}{2}, 4\right) \)  
(d) \( (0, \frac{3}{2}) \)  
14. \( g(x, y) = \int_x^1 \frac{1}{t} \, dt \)  
(a) \( (4, 1) \)  
(b) \( (6, 3) \)  
(c) \( (2, 5) \)  
(d) \( \left(\frac{3}{2}, 7\right) \)  
15. \( f(x, y) = x^2 - 2y \)  
16. \( f(x, y) = 3xy + y^2 \)  
(a) \( \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \)  
(b) \( \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \)  
17. \( f(x, y) = \sqrt{4 - x^2 - y^2} \)  
18. \( f(x, y) = \sqrt{4 - x^2 - 4y^2} \)  
19. \( f(x, y) = \arcsin(x + y) \)  
20. \( f(x, y) = \arccos(y/x) \)

In Exercises 17–28, describe the domain and range of the function.

21. \( f(x, y) = \ln(4 - x - y) \)  
22. \( f(x, y) = \ln(xy - 6) \)  
23. \( z = \frac{x + y}{xy} \)  
24. \( z = \frac{xy}{x - y} \)  
25. \( f(x, y) = e^{x/y} \)  
26. \( f(x, y) = x^2 + y^2 \)  
27. \( g(x, y) = \frac{1}{xy} \)  
28. \( g(x, y) = x \sqrt{y} \)  
29. Think About It The graphs labeled (a), (b), (c), and (d) are graphs of the function \( f(x, y) = -4x/(x^2 + y^2 + 1) \). Match the four graphs with the points in space from which the surface is viewed. The four points are \((20, 15, 25), (-15, 10, 20), (20, 20, 0), \) and \((20, 0, 0)\).

(a) \( \text{Graphic (a)} \)  
(b) \( \text{Graphic (b)} \)  
(c) \( \text{Graphic (c)} \)  
(d) \( \text{Graphic (d)} \)  
30. Think About It Use the function given in Exercise 29.

(a) Find the domain and range of the function.

(b) Identify the points in the \( xy \)-plane where the function value is 0.

(c) Does the surface pass through all the octants of the rectangular coordinate system? Give reasons for your answer.

In Exercises 31–38, sketch the surface given by the function.

31. \( f(x, y) = 5 \)  
32. \( f(x, y) = 6 - 2x - 3y \)  
33. \( f(x, y) = y^2 \)  
34. \( g(x, y) = \frac{1}{2}x \)  
35. \( z = 4 - x^2 - y^2 \)  
36. \( z = \frac{1}{4}x^2 + y^2 \)  
37. \( f(x, y) = e^{-x} \)  
38. \( f(x, y) = \begin{cases} xy, & \text{if } x \geq 0, y \geq 0 \\ 0, & \text{if } x < 0 \text{ or } y < 0 \end{cases} \)

In Exercises 39–42, use a computer algebra system to graph the function.

39. \( z = y^2 - x^2 + 1 \)  
40. \( z = \frac{1}{16} \sqrt{144 - 16x^2 - 9y^2} \)  
41. \( f(x, y) = x^2 e^{-y/2} \)  
42. \( f(x, y) = x \sin y \)
43. Conjecture Consider the function \( f(x, y) = x^2 + y^2 \).
(a) Sketch the graph of the surface given by \( f \).
(b) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = f(x, y) + 2 \). Use a computer algebra system to confirm your answer.
(c) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = f(x, y) - 2 \). Use a computer algebra system to confirm your answer.
(d) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = 4 - f(x, y) \). Use a computer algebra system to confirm your answer.
(e) On the surface in part (a), sketch the graphs of \( z = f(1, y) \) and \( z = f(x, 1) \).

44. Conjecture Consider the function \( f(x, y) = xy \), for \( x \geq 0 \) and \( y \geq 0 \).
(a) Sketch the graph of the surface given by \( f \).
(b) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = f(x, y) - 3 \). Use a computer algebra system to confirm your answer.
(c) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = -f(x, y) \). Use a computer algebra system to confirm your answer.
(d) Make a conjecture about the relationship between the graphs of \( f \) and \( g(x, y) = \frac{1}{2}f(x, y) \). Use a computer algebra system to confirm your answer.
(e) On the surface in part (a), sketch the graph of \( z = f(x, y) \).

In Exercises 45–48, match the graph of the surface with one of the contour maps. [The contour maps are labeled (a), (b), (c), and (d).]

(a)  
(b)  
(c)  
(d)  

45. \( f(x, y) = e^{1-x^2-y^2} \)
46. \( f(x, y) = e^{1-x^2+y^2} \)

47. \( f(x, y) = \ln|y - x^2| \)
48. \( f(x, y) = \cos\left(\frac{x^2 + 2y^2}{4}\right) \)

In Exercises 49–56, describe the level curves of the function. Sketch the level curves for the given \( c \)-values.
49. \( z = x + y \), \( c = -1, 0, 2, 4 \)
50. \( z = 6 - 2x - 3y \), \( c = 0, 2, 4, 6, 8, 10 \)
51. \( z = \sqrt{25 - x^2 - y^2} \), \( c = 0, 1, 2, 3, 4, 5 \)
52. \( f(x, y) = x^2 + 2y^2 \), \( c = 0, 2, 4, 6, 8 \)
53. \( f(x, y) = xy \), \( c = \pm 1, \pm 2, \ldots, \pm 6 \)
54. \( f(x, y) = e^{x/2} \), \( c = 2, 3, 4, 5, 6, 7, 8 \)
55. \( f(x, y) = \frac{x}{x^2 + y^2} \), \( c = \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2 \)
56. \( f(x, y) = \ln(x - y) \), \( c = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2 \)

In Exercises 57–60, use a graphing utility to graph six level curves of the function.
57. \( f(x, y) = x^2 - y^2 + 2 \)
58. \( f(x, y) = |xy| \)
59. \( g(x, y) = \frac{8}{1 + x^2 + y^2} \)
60. \( h(x, y) = 3\sin(|x| + |y|) \)

Writing About Concepts

61. Define a function of two variables.
62. What is a graph of a function of two variables? How is it interpreted geometrically? Describe level curves.
63. All of the level curves of the surface given by \( z = f(x, y) \) are concentric circles. Does this imply that the graph of \( f \) is a hemisphere? Illustrate your answer with an example.
64. Construct a function whose level curves are lines passing through the origin.
Writing In Exercises 65 and 66, use the graphs of the level curves (c-values evenly spaced) of the function \( f \) to write a description of a possible graph of \( f \). Is the graph of \( f \) unique? Explain.

65.  

66.  

67. Investment In 2005, an investment of $1000 was made in a bond earning 10% compounded annually. Assume that the buyer pays tax at rate \( R \) and the annual rate of inflation is \( I \). In the year 2015, the value \( V \) of the investment in constant 2005 dollars is

\[
V(I, R) = 1000 \left[ \frac{1 + 0.10(1 - R)}{1 + I} \right]^{10}.
\]

Use this function of two variables to complete the table.

<table>
<thead>
<tr>
<th>Tax Rate</th>
<th>Inflation Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.03</td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0.28</td>
<td></td>
</tr>
<tr>
<td>0.35</td>
<td></td>
</tr>
</tbody>
</table>

68. Investment A principal of $1000 is deposited in a savings account that earns an interest rate of \( r \) (written as a decimal), compounded continuously. The amount \( A(r, t) \) after \( t \) years is

\[
A(r, t) = 1000e^{rt}.
\]

Use this function of two variables to complete the table.

<table>
<thead>
<tr>
<th>Number of Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate</td>
</tr>
<tr>
<td>0.02</td>
</tr>
<tr>
<td>0.04</td>
</tr>
<tr>
<td>0.06</td>
</tr>
<tr>
<td>0.08</td>
</tr>
</tbody>
</table>

In Exercises 69–74, sketch the graph of the level surface \( f(x, y, z) = c \) at the given value of \( c \).

69. \( f(x, y, z) = x - 2y + 3z, \ c = 6 \)

70. \( f(x, y, z) = 4x + y + 2z, \ c = 4 \)

71. \( f(x, y, z) = x^2 + y^2 + z^2, \ c = 9 \)

72. \( f(x, y, z) = x^2 + \frac{1}{4}y^2 - z, \ c = 1 \)

73. \( f(x, y, z) = 4x^2 + 4y^2 - z^2, \ c = 0 \)

74. \( f(x, y, z) = \sin x - z, \ c = 0 \)

75. Forestry The Doyle Log Rule is one of several methods used to determine the lumber yield of a log (in board-feet) in terms of its diameter \( d \) (in inches) and its length \( L \) (in feet). The number of board-feet is

\[
N(d, L) = \frac{(d - 4)^2}{4} L.
\]

(a) Find the number of board-feet of lumber in a log 22 inches in diameter and 12 feet in length.

(b) Find \( N(30, 12) \).

76. Queuing Model The average length of time that a customer waits in line for service is

\[
W(x, y) = \frac{1}{x - y}, \quad x > y
\]

where \( y \) is the average arrival rate, written as the number of customers per unit of time, and \( x \) is the average service rate written in the same units. Evaluate each of the following.

(a) \( W(15, 10) \)  
(b) \( W(12, 9) \)  
(c) \( W(12, 6) \)  
(d) \( W(4, 2) \)

77. Temperature Distribution The temperature \( T \) (in degrees Celsius) at any point \((x, y)\) in a circular steel plate of radius 10 meters is

\[
T = 600 - 0.75x^2 - 0.75y^2
\]

where \( x \) and \( y \) are measured in meters. Sketch some of the isothermal curves.

78. Electric Potential The electric potential \( V \) at any point \((x, y)\) is

\[
V(x, y) = \frac{5}{\sqrt{25 + x^2 + y^2}}
\]

Sketch the equipotential curves for \( V = \frac{1}{2}, V = \frac{1}{3}, \) and \( V = \frac{1}{4} \).

79. Cobb-Douglas Production Function Use the Cobb-Douglas production function (see Example 5) to show that if the number of units of labor and the number of units of capital are doubled the production level is also doubled.

80. Cobb-Douglas Production Function Show that the Cobb-Douglas production function \( z = Cx^a y^{1-a} \) can be rewritten as

\[
\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}
\]

81. Construction Cost A rectangular box with an open top has length of \( x \) feet, a width of \( y \) feet, and a height of \( z \) feet. It costs $0.75 per square foot to build the base and $0.40 per square foot to build the sides. Write the cost \( C \) of constructing the box as a function of \( x, y, \) and \( z \).

82. Volume A propane tank is constructed by welding hemispheres to the ends of a right circular cylinder. Write the volume \( V \) of the tank as a function of \( r \) and \( l \), where \( r \) is the radius of the cylinder and hemispheres, and \( l \) is the length of the cylinder.
83. **Ideal Gas Law** According to the Ideal Gas Law, \( PV = kT \), where \( P \) is pressure, \( V \) is volume, \( T \) is temperature (in Kelvins), and \( k \) is a constant of proportionality. A tank contains 2600 cubic inches of nitrogen at a pressure of 20 pounds per square inch and a temperature of 300 K.

(a) Determine \( k \).
(b) Write \( P \) as a function of \( V \) and \( T \) and describe the level curves.

84. **Modeling Data** The table shows the net sales \( x \) (in billions of dollars), the total assets \( y \) (in billions of dollars), and the shareholder’s equity \( z \) (in billions of dollars) for Wal-Mart for the years 1998 through 2003. (Source: 2003 Annual Report for Wal-Mart)

<table>
<thead>
<tr>
<th>Year</th>
<th>1998</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>118.0</td>
<td>137.6</td>
<td>165.0</td>
<td>191.3</td>
<td>217.8</td>
<td>244.5</td>
</tr>
<tr>
<td>( y )</td>
<td>45.4</td>
<td>50.0</td>
<td>70.3</td>
<td>78.1</td>
<td>83.5</td>
<td>94.7</td>
</tr>
<tr>
<td>( z )</td>
<td>18.5</td>
<td>21.1</td>
<td>25.8</td>
<td>31.3</td>
<td>35.1</td>
<td>39.3</td>
</tr>
</tbody>
</table>

A model for these data is
\[ z = f(x, y) = 0.156x + 0.031y - 1.66. \]

(a) Use a graphing utility and the model to approximate for \( z \) for the given values of \( x \) and \( y \).
(b) Which of the two variables in this model has the greater influence on shareholder’s equity?
(c) Simplify the expression for \( f(x, 55) \) and interpret its meaning in the context of the problem.

85. **Meteorology** Meteorologists measure the atmospheric pressure in millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). On the map, the closer the isobars the higher the wind speed. Match points \( A, B, \) and \( C \) with (a) highest pressure, (b) lowest pressure, and (c) highest wind velocity.

86. **Acid Rain** The acidity of rainwater is measured in units called pH. A pH of 7 is neutral, smaller values are increasingly acidic, and larger values are increasingly alkaline. The map shows the curves of equal pH and gives evidence that downwind of heavily industrialized areas the acidity has been increasing. Using the level curves on the map, determine the direction of the prevailing winds in the northeastern United States.

87. **Air Conditioner Use** The contour map shown in the figure represents the estimated annual hours of air conditioner use for an average household. (Source: Association of Home Appliance Manufacturers)

(a) Discuss the use of color to represent the level curves.
(b) Do the level curves correspond to equally spaced annual usage hours? Explain.
(c) Describe how to obtain a more detailed contour map.

88. **Geology** The contour map in the figure represents color-coded seismic amplitudes of a fault horizon and a projected contour map, which is used in earthquake studies. (Source: Adapted from Shipman/Wilson/Todd, An Introduction to Physical Science, Eighth Edition)

(a) Discuss the use of color to represent the level curves.
(b) Do the level curves correspond to equally spaced amplitudes? Explain.

**True or False?** In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

89. If \( f(x_0, y_0) = f(x_1, y_1) \), then \( x_0 = x_1 \) and \( y_0 = y_1 \).
90. A vertical line can intersect the graph of \( z = f(x, y) \) at most once.
91. If \( f \) is a function, then \( f(ax, ay) = af(x, y) \).
92. The graph of \( f(x, y) = x^2 - y^2 \) is a hyperbolic paraboloid.
SONYA KOVALEVSKY (1850–1891)
Much of the terminology used to define limits and continuity of a function of two or three variables was introduced by the German mathematician Karl Weierstrass (1815–1897). Weierstrass’s rigorous approach to limits and other topics in calculus gained him the reputation as the “father of modern analysis.” Weierstrass was a gifted teacher. One of his best-known students was the Russian mathematician Sonya Kovalevsky, who applied many of Weierstrass’s techniques to problems in mathematical physics and became one of the first women to gain acceptance as a research mathematician.

SONYA KOVALEVSKY

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896  CHAPTER 13  Functions of Several Variables
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**Section 13.2**

**Limits and Continuity**

- Understand the definition of a neighborhood in the plane.
- Understand and use the definition of the limit of a function of two variables.
- Extend the concept of continuity to a function of two variables.
- Extend the concept of continuity to a function of three variables.

**Neighborhoods in the Plane**

In this section, you will study limits and continuity involving functions of two or three variables. The section begins with functions of two variables. At the end of the section, the concepts are extended to functions of three variables.

We begin our discussion of the limit of a function of two variables by defining a two-dimensional analog to an interval on the real line. Using the formula for the distance between two points and in the plane, you can define the \( \delta \)-neighborhood about \((x_0, y_0)\) to be the disk centered at \((x_0, y_0)\) with radius \(\delta > 0\) as shown in Figure 13.18. When this formula contains the less than inequality, \(<\), the disk is called open, and when it contains the less than or equal to inequality, \(\leq\), the disk is called closed. This corresponds to the use of \(<\) and \(\leq\) to define open and closed intervals.

```
\[ \{ (x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \} \]
```

An open disk

Figure 13.18

The boundary and interior points of a region \(R\)

Figure 13.19

A point \((x_0, y_0)\) in a plane region \(R\) is an interior point of \(R\) if there exists a \(\delta\)-neighborhood about \((x_0, y_0)\) that lies entirely in \(R\), as shown in Figure 13.19. If every point in \(R\) is an interior point, then \(R\) is an open region. A point \((x_0, y_0)\) is a boundary point of \(R\) if every open disk centered at \((x_0, y_0)\) contains points inside \(R\) and points outside \(R\). By definition, a region must contain its interior points, but it need not contain its boundary points. If a region contains all its boundary points, the region is closed. A region that contains some but not all of its boundary points is neither open nor closed.

**FOR FURTHER INFORMATION** For more information on Sonya Kovalevsky, see the article “S. Kovalevsky: A Mathematical Lesson” by Karen D. Rappaport in *The American Mathematical Monthly*. 
**Limit of a Function of Two Variables**

**Definition of the Limit of a Function of Two Variables**

Let \( f \) be a function of two variables defined, except possibly at \((x_0, y_0)\), on an open disk centered at \((x_0, y_0)\), and let \( L \) be a real number. Then

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L
\]

if for each \( \varepsilon > 0 \) there corresponds a \( \delta > 0 \) such that

\[
|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.
\]

NOTE  Graphically, this definition of a limit implies that for any point \((x, y) \neq (x_0, y_0)\) in the disk of radius \( \delta \), the value \( f(x, y) \) lies between \( L + \varepsilon \) and \( L - \varepsilon \), as shown in Figure 13.20.

The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference. To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left. If the function approaches the same limit from the right and from the left, you can conclude that the limit exists. However, for a function of two variables, the statement

\[
(x, y) \to (x_0, y_0)
\]

means that the point \((x, y)\) is allowed to approach \((x_0, y_0)\) from any direction. If the value of

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y)
\]

is not the same for all possible approaches, or **paths**, to \((x_0, y_0)\), the limit does not exist.

**EXAMPLE 1  Verifying a Limit by the Definition**

Show that

\[
\lim_{(x, y) \to (a, b)} x = a.
\]

**Solution**  Let \( f(x, y) = x \) and \( L = a \). You need to show that for each \( \varepsilon > 0 \), there exists a \( \delta \)-neighborhood about \((a, b)\) such that

\[
|f(x, y) - L| = |x - a| < \varepsilon
\]

whenever \((x, y) \neq (a, b)\) lies in the neighborhood. You can first observe that from

\[
0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta
\]

it follows that

\[
|f(x, y) - a| = |x - a| = \sqrt{(x - a)^2} \leq \sqrt{(x - a)^2 + (y - b)^2} < \delta.
\]

So, you can choose \( \delta = \varepsilon \), and the limit is verified.
Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables. (See Theorem 1.2 in Section 1.3.) Some of these properties are used in the next example.

**EXAMPLE 2  Verifying a Limit**

Evaluate \( \lim_{(x,y) \to (1,2)} \frac{5x^2y}{x^2 + y^2} \).

**Solution**  By using the properties of limits of products and sums, you obtain

\[
\lim_{(x,y) \to (1,2)} 5x^2y = 5(1^2)(2) = 10
\]

and

\[
\lim_{(x,y) \to (1,2)} (x^2 + y^2) = (1^2 + 2^2) = 5.
\]

Because the limit of a quotient is equal to the quotient of the limits (and the denominator is not 0), you have

\[
\lim_{(x,y) \to (1,2)} \frac{5x^2y}{x^2 + y^2} = \frac{10}{5} = 2.
\]

**Try It**  **Exploration A**

**EXAMPLE 3  Verifying a Limit**

Evaluate \( \lim_{(x,y) \to (0,0)} \frac{5x^2y}{x^2 + y^2} \).

**Solution**  In this case, the limits of the numerator and of the denominator are both 0, and so you cannot determine the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing. However, from the graph of \( f \) in Figure 13.21, it seems reasonable that the limit might be 0. So, you can try applying the definition to \( L = 0 \). First, note that

\[
|y| \leq \sqrt{x^2 + y^2} \quad \text{and} \quad \frac{x^2}{x^2 + y^2} \leq 1.
\]

Then, in a \( \delta \)-neighborhood about \((0,0)\), you have \( 0 < \sqrt{x^2 + y^2} < \delta \), and it follows that, for \((x,y) \neq (0,0)\),

\[
|f(x,y) - 0| = \left| \frac{5x^2y}{x^2 + y^2} \right| = 5|y| \left( \frac{x^2}{x^2 + y^2} \right) \leq 5|y| \leq 5 \sqrt{x^2 + y^2} < 5\delta.
\]

So, you can choose \( \delta = \varepsilon/5 \) and conclude that

\[
\lim_{(x,y) \to (0,0)} \frac{5x^2y}{x^2 + y^2} = 0.
\]

**Try It**  **Exploration A**
For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit
\[
\lim_{(x, y) \to (0, 0)} \frac{1}{x^2 + y^2}
\]
does not exist because the values of \(f(x, y)\) increase without bound as \((x, y)\) approaches \((0, 0)\) along any path (see Figure 13.22).

For other functions, it is not so easy to recognize that a limit does not exist. For instance, the next example describes a limit that does not exist because the function values of increase without bound as \((x, y)\) approaches \((0, 0)\).

**EXAMPLE 4  A Limit That Does Not Exist**

Show that the following limit does not exist.
\[
\lim_{(x, y) \to (0, 0)} \frac{(x^2 - y^2)^2}{x^2 + y^2}
\]

**Solution**
The domain of the function given by
\[
f(x, y) = \frac{(x^2 - y^2)^2}{x^2 + y^2}
\]
consists of all points in the \(xy\)-plane except for the point \((0, 0)\). To show that the limit as \((x, y)\) approaches \((0, 0)\) does not exist, consider approaching \((0, 0)\) along two different “paths,” as shown in Figure 13.23. Along the \(x\)-axis, every point is of the form \((x, 0)\), and the limit along this approach is
\[
\lim_{(x, 0) \to (0, 0)} \frac{(x^2 - 0^2)^2}{x^2 + 0^2} = \lim_{(x, 0) \to (0, 0)} 1^2 = 1. \quad \text{Limit along } x\text{-axis}
\]
However, if \((x, y)\) approaches \((0, 0)\) along the line \(y = x\), you obtain
\[
\lim_{(x, x) \to (0, 0)} \frac{(x^2 - x^2)^2}{x^2 + x^2} = \lim_{(x, x) \to (0, 0)} \left( \frac{0}{2x^2} \right)^2 = 0. \quad \text{Limit along line } y = x
\]
This means that in any open disk centered at \((0, 0)\) there are points \((x, y)\) at which \(f\) takes on the value 1, and other points at which \(f\) takes on the value 0. For instance, \(f(x, y) = 1\) at the points \((1, 0), (0.1, 0), (0.01, 0)\), and \((0.001, 0)\) and \(f(x, y) = 0\) at the points \((1, 1), (0.1, 0.1), (0.01, 0.01)\), and \((0.001, 0.001)\). So, \(f\) does not have a limit as \((x, y) \to (0, 0)\).
Continuity of a Function of Two Variables

Notice in Example 2 that the limit of \( f(x, y) = 5x^2y/(x^2 + y^2) \) as \((x, y) \to (1, 2)\) can be evaluated by direct substitution. That is, the limit is \( f(1, 2) = 2 \). In such cases the function \( f \) is said to be **continuous** at the point \((1, 2)\).

**Definition of Continuity of a Function of Two Variables**

A function \( f \) of two variables is **continuous at a point** \((x_0, y_0)\) in an open region \( R \) if \( f(x_0, y_0) \) is equal to the limit of \( f(x, y) \) as \((x, y)\) approaches \((x_0, y_0)\). That is,

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0).
\]

The function \( f \) is **continuous in the open region** \( R \) if it is continuous at every point in \( R \).

In Example 3, it was shown that the function

\[
f(x, y) = \frac{5x^2y}{x^2 + y^2}
\]

is not continuous at \((0, 0)\). However, because the limit at this point exists, you can remove the discontinuity by defining \( f \) at \((0, 0)\) as being equal to its limit there. Such a discontinuity is called **removable**. In Example 4, the function

\[
f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2
\]

was also shown not to be continuous at \((0, 0)\), but this discontinuity is **nonremovable**.

**Theorem 13.1 Continuous Functions of Two Variables**

If \( k \) is a real number and \( f \) and \( g \) are continuous at \((x_0, y_0)\), then the following functions are continuous at \((x_0, y_0)\).

1. Scalar multiple: \( kf \)
2. Sum and difference: \( f \pm g \)
3. Product: \( fg \)
4. Quotient: \( f/g \), if \( g(x_0, y_0) \neq 0 \)

Theorem 13.1 establishes the continuity of **polynomial** and **rational** functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables. For instance, the functions whose graphs are shown in Figures 13.24 and 13.25 are continuous at every point in the plane.
The next theorem states conditions under which a composite function is continuous.

**THEOREM 13.2 Continuity of a Composite Function**

If \( h \) is continuous at \((x_0, y_0)\) and \( g \) is continuous at \( h(x_0, y_0) \), then the composite function given by \((g \circ h)(x, y) = g(h(x, y))\) is continuous at \((x_0, y_0)\). That is,

\[
\lim_{(x, y) \to (x_0, y_0)} g(h(x, y)) = g(h(x_0, y_0)).
\]

**NOTE** Note in Theorem 13.2 that \( h \) is a function of two variables and \( g \) is a function of one variable.

**EXAMPLE 5 Testing for Continuity**

Discuss the continuity of each function.

a. \( f(x, y) = \frac{x - 2y}{x^2 + y^2} \)

b. \( g(x, y) = \frac{2}{y - x^2} \)

**Solution**

a. Because a rational function is continuous at every point in its domain, you can conclude that \( f \) is continuous at each point in the \( xy \)-plane except at \((0, 0)\), as shown in Figure 13.26.

b. The function given by \( g(x, y) = \frac{2}{y - x^2} \) is continuous except at the points at which the denominator is 0, \( y - x^2 = 0 \). So, you can conclude that the function is continuous at all points except those lying on the parabola \( y = x^2 \). Inside this parabola, you have \( y > x^2 \), and the surface represented by the function lies above the \( xy \)-plane, as shown in Figure 13.27. Outside the parabola, \( y < x^2 \), and the surface lies below the \( xy \)-plane.
Continuity of a Function of Three Variables

The preceding definitions of limits and continuity can be extended to functions of three variables by considering points \((x, y, z)\) within the open sphere

\[
(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2.
\]

The radius of this sphere is \(\delta\), and the sphere is centered at \((x_0, y_0, z_0)\), as shown in Figure 13.28. A point \((x_0, y_0, z_0)\) in a region \(R\) in space is an interior point of \(R\) if there exists a \(\delta\)-sphere about \((x_0, y_0, z_0)\) that lies entirely in \(R\). If every point in \(R\) is an interior point, then \(R\) is called open.

**Definition of Continuity of a Function of Three Variables**

A function \(f\) of three variables is continuous at a point \((x_0, y_0, z_0)\) in an open region \(R\) if \(f(x_0, y_0, z_0)\) is defined and is equal to the limit of \(f(x, y, z)\) as \((x, y, z)\) approaches \((x_0, y_0, z_0)\). That is,

\[
\lim_{(x, y, z) \to (x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0).
\]

The function \(f\) is continuous in the open region \(R\) if it is continuous at every point in \(R\).

**EXAMPLE 6** Testing Continuity of a Function of Three Variables

The function

\[
f(x, y, z) = \frac{1}{x^2 + y^2 - z}
\]

is continuous at each point in space except at the points on the paraboloid given by \(z = x^2 + y^2\).
Exercises for Section 13.2

The symbol ♻ indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on ♻ to view the complete solution of the exercise.

Click on ♻ to print an enlarged copy of the graph.

In Exercises 1–4, use the definition of the limit of a function of two variables to verify the limit.

1. \[ \lim_{(x, y) \to (2, 3)} x = 2 \]
2. \[ \lim_{(x, y) \to (4, -1)} x = 4 \]
3. \[ \lim_{(x, y) \to (1, -3)} y = -3 \]
4. \[ \lim_{(x, y) \to (a, b)} y = b \]

In Exercises 5–8, find the indicated limit by using the limits

\[ \lim_{(x, y) \to (a, b)} f(x, y) = 5 \text{ and } \lim_{(x, y) \to (a, b)} g(x, y) = 3. \]

5. \[ \lim_{(x, y) \to (a, b)} [f(x, y) - g(x, y)] \]
6. \[ \lim_{(x, y) \to (a, b)} \left[ \frac{4f(x, y)}{g(x, y)} \right] \]
7. \[ \lim_{(x, y) \to (a, b)} [f(x, y)g(x, y)] \]
8. \[ \lim_{(x, y) \to (a, b)} \left[ \frac{f(x, y) - g(x, y)}{f(x, y)} \right] \]

In Exercises 9–18, find the limit and discuss the continuity of the function.

9. \[ \lim_{(x, y) \to (2, 1)} (x + 3y^2) \]
10. \[ \lim_{(x, y) \to (0, 0)} (5x + y + 1) \]
11. \[ \lim_{(x, y) \to (2, 4)} \frac{x + y}{x - y} \]
12. \[ \lim_{(x, y) \to (1, 1)} \frac{x}{\sqrt{x + y}} \]
13. \[ \lim_{(x, y) \to (0, 1)} \arcsin(x/y) \]
14. \[ \lim_{(x, y) \to (4/2, 2)} y \cos xy \]
15. \[ \lim_{(x, y) \to (-1, 2)} e^{xy} \]
16. \[ \lim_{(x, y) \to (1, 1)} \frac{xy}{x^2 + y^2} \]
17. \[ \lim_{(x, y, z) \to (1, 2, 5)} \sqrt{x + y + z} \]
18. \[ \lim_{(x, y, z) \to (2, 0, 1)} x e^{yz} \]
In Exercises 19–24, find the limit (if it exists). If the limit does not exist, explain why.

19. \[ \lim_{(x, y) \to (0, 0)} \frac{x + y}{x^2 + y} \]
20. \[ \lim_{(x, y) \to (0, 0)} \frac{x}{x^2 - y^2} \]
21. \[ \lim_{(x, y) \to (1, 1)} \frac{xy - 1}{1 + xy} \]
22. \[ \lim_{(x, y) \to (0, 0)} \frac{x + y}{x + y^3} \]

In Exercises 25–28, discuss the continuity of the function and evaluate the limit of \( f(x, y) \) (if it exists) as \((x, y) \to (0, 0)\).

25. \[ f(x, y) = e^{xy} \]

26. \[ f(x, y) = \frac{x^2}{(x^2 + 1)(y^2 + 1)} \]

27. \[ f(x, y) = \ln(x^2 + y^2) \]

28. \[ f(x, y) = 1 - \frac{\cos(x^2 + y^2)}{x^2 + y^2} \]

In Exercises 29–32, use a graphing utility to make a table showing the values of \( f(x, y) \) at the given points. Use the result to make a conjecture about the limit of \( f(x, y) \) as \((x, y) \to (0, 0)\).

Determine whether the limit exists analytically and discuss the continuity of the function.

29. \[ f(x, y) = \frac{xy}{x^2 + y^2} \]
   - Path: \( y = 0 \)
   - Points: \( (1, 0), (0.5, 0), (0.1, 0), (0.01, 0), (0.001, 0) \)
   - Path: \( y = x \)
   - Points: \( (1, 1), (0.5, 0.5), (0.1, 0.1), (0.01, 0.01), (0.001, 0.001) \)

30. \[ f(x, y) = \frac{y}{x^2 + y^2} \]
   - Path: \( y = 0 \)
   - Points: \( (1, 0), (0.5, 0), (0.1, 0), (0.01, 0), (0.001, 0) \)
   - Path: \( y = x \)
   - Points: \( (1, 1), (0.5, 0.5), (0.1, 0.1), (0.01, 0.01), (0.001, 0.001) \)

31. \[ f(x, y) = -\frac{xy^2}{x^2 + y^2} \]
   - Path: \( x = y^2 \)
   - Points: \( (1, 1), (0.25, 0.5), (0.01, 0.1), (0.0001, 0.01), (0.000001, 0.001) \)
   - Path: \( x = -y^2 \)
   - Points: \( (-1, 1), (-0.25, 0.5), (-0.01, 0.1), (-0.0001, 0.01), (-0.000001, 0.001) \)
In Exercises 41–48, use polar coordinates to find the limit.

41. \( \lim_{(x, y) \to (0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \)

42. \( \lim_{(x, y) \to (0, 0)} \frac{xy^2}{x^2 + y^2} \)

43. \( \lim_{(x, y) \to (0, 0)} \frac{x^3 + y^3}{x^2 + y^2} \)

44. \( \lim_{(x, y) \to (0, 0)} \frac{x^2y^2}{x^2 + y^2} \)

45. \( \lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \)

46. \( \lim_{(x, y) \to (0, 0)} \frac{\sin \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} \)

47. \( \lim_{(x, y) \to (0, 0)} (x^2 + y^2)^{\ln(x^2 + y^2)} \)

48. \( \lim_{(x, y) \to (0, 0)} \frac{1 - \cos(x^2 + y^2)}{x^2 + y^2} \)

In Exercises 49–54, discuss the continuity of the function.

49. \( f(x, y, z) = \frac{1}{x^2 + y^2 + z^2} \)

50. \( f(x, y, z) = \frac{z}{x^2 + y^2 - 9} \)

51. \( f(x, y, z) = \frac{\sin z}{e^{x^2} + e^y} \)

52. \( f(x, y, z) = xy \sin z \)

53. \( f(x, y) = \begin{cases} \sin xy, & xy \neq 0 \\ x, & xy = 0 \end{cases} \)

54. \( f(x, y) = \begin{cases} \sin(x^2 - y^2), & x^2 \neq y^2 \\ 1, & x^2 = y^2 \end{cases} \)

In Exercises 55–58, discuss the continuity of the composite function \( f \circ g \).

55. \( f(t) = t^2 \
\quad g(x, y) = 3x - 2y \)

56. \( f(t) = \frac{1}{t} \
\quad g(x, y) = x^2 + y^2 \)

57. \( f(t) = \frac{1}{t} \
\quad g(x, y) = 3x - 2y \)

58. \( f(t) = \frac{1}{4 - t} \
\quad g(x, y) = x^2 + y^2 \)

In Exercises 59–62, find each limit.

(a) \( \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \)

(b) \( \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \)

59. \( f(x, y) = x^2 - 4y \)

60. \( f(x, y) = x^2 + y^2 \)

61. \( f(x, y) = 2x + xy - 3y \)

62. \( f(x, y) = \sqrt{y} (y + 1) \)
True or False? In Exercises 63–66, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

63. If \( \lim_{{(x, y) \to (0, 0)}} f(x, y) = 0 \), then \( \lim_{{x \to 0}} f(x, 0) = 0 \).

64. If \( \lim_{{(x, y) \to (0, 0)}} f(0, y) = 0 \), then \( \lim_{{y \to 0}} f(x, y) = 0 \).

65. If \( f \) is continuous for all nonzero \( x \) and \( y \), and \( f(0, 0) = 0 \), then \( \lim_{{(x, y) \to (0, 0)}} f(x, y) = 0 \).

66. If \( g \) and \( h \) are continuous functions of \( x \) and \( y \), and \( f(x, y) = g(x) + h(y) \), then \( f \) is continuous.

Writing About Concepts

67. Define the limit of a function of two variables. Describe a method for showing that

\[
\lim_{{(x, y) \to (a, b)}} f(x, y)
\]
does not exist.

68. State the definition of continuity of a function of two variables.

69. If \( f(2, 3) = 4 \), can you conclude anything about

\[
\lim_{{(x, y) \to (2, 3)}} f(x, y)
\]

Give reasons for your answer.

70. If \( \lim_{{(x, y) \to (2, 3)}} f(x, y) = 4 \), can you conclude anything about \( f(2, 3) \)? Give reasons for your answer.

71. Consider \( \lim_{{(x, y) \to (0, 0)}} \frac{x^2 + y^2}{xy} \) (see figure).

(a) Determine (if possible) the limit along any line of the form \( y = ax \).

(b) Determine (if possible) the limit along the parabola \( y = x^2 \).

(c) Does the limit exist? Explain.

72. Consider \( \lim_{{(x, y) \to (0, 0)}} \frac{x^2 y}{x^4 + y^2} \) (see figure).

(a) Determine (if possible) the limit along any line of the form \( y = ax \).

(b) Determine (if possible) the limit along the parabola \( y = x^2 \).

(c) Does the limit exist? Explain.

In Exercises 73 and 74, use spherical coordinates to find the limit. \( \text{[Hint: Let } x = \rho \sin \phi \cos \theta, \ y = \rho \sin \phi \sin \theta, \ \text{and } z = \rho \cos \phi, \ \text{and note that } (x, y, z) \to (0, 0, 0) \text{ implies } \rho \to 0^+.\text{]} \)

73. \( \lim_{{(x, y, z) \to (0, 0, 0)}} \frac{xyz}{x^2 + y^2 + z^2} \)

74. \( \lim_{{(x, y, z) \to (0, 0, 0)}} \tan^{-1} \left[ \frac{1}{x^2 + y^2 + z^2} \right] \)

75. Find the following limit.

\( \lim_{{(x, y) \to (0, 1)}} \tan^{-1} \left[ \frac{x^2 + 1}{x^2 + (y - 1)^2} \right] \)

76. For the function

\[ f(x, y) = xy \left( \frac{x^2 - y^2}{x^2 + y^2} \right) \]

define \( f(0, 0) \) such that \( f \) is continuous at the origin.

77. Prove that

\[ \lim_{{(x, y) \to (a, b)}} [f(x, y) + g(x, y)] = L_1 + L_2 \]

where \( f(x, y) \to L_1 \) and \( g(x, y) \to L_2 \) as \( (x, y) \to (a, b) \).

78. Prove that if \( f \) is continuous and \( f(a, b) < 0 \), there exists a \( \delta \)-neighborhood about \( (a, b) \) such that \( f(x, y) < 0 \) for every point \( (x, y) \) in the neighborhood.
Partial Derivatives

- Find and use partial derivatives of a function of two variables.
- Find and use partial derivatives of a function of three or more variables.
- Find higher-order partial derivatives of a function of two or three variables.

Partial Derivatives of a Function of Two Variables

In applications of functions of several variables, the question often arises, “How will the value of a function be affected by a change in one of its independent variables?” You can answer this by considering the independent variables one at a time. For example, to determine the effect of a catalyst in an experiment, a chemist could conduct the experiment several times using varying amounts of the catalyst, while keeping constant other variables such as temperature and pressure. You can use a similar procedure to determine the rate of change of a function with respect to one of its several independent variables. This process is called partial differentiation, and the result is referred to as the partial derivative of with respect to the chosen independent variable.

**Definition of Partial Derivatives of a Function of Two Variables**

If \( z = f(x, y) \), then the first partial derivatives of \( f \) with respect to \( x \) and \( y \) are the functions \( f_x \) and \( f_y \) defined by

\[
\begin{align*}
  f_x(x, y) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\
  f_y(x, y) &= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}
\end{align*}
\]

provided the limits exist.

This definition indicates that if \( z = f(x, y) \), then to find \( f_x \), you consider \( y \) constant and differentiate with respect to \( x \). Similarly, to find \( f_y \), you consider \( x \) constant and differentiate with respect to \( y \).

**EXAMPLE 1** Finding Partial Derivatives

Find the partial derivatives \( f_x \) and \( f_y \) for the function

\[
f(x, y) = 3x - x^2y^2 + 2x^3y. \quad \text{Original function}
\]

**Solution**

Considering \( y \) to be constant and differentiating with respect to \( x \) produces

\[
\begin{align*}
  f_x(x, y) &= 3 - 2xy^2 + 6x^2y. \quad \text{Partial derivative with respect to } x \\
  f_y(x, y) &= -2x^2y + 2x^3. \quad \text{Partial derivative with respect to } y
\end{align*}
\]
**Notation for First Partial Derivatives**

For \( z = f(x, y) \), the partial derivatives \( f_x \) and \( f_y \) are denoted by

\[
\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}
\]

and

\[
\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}
\]

The first partials evaluated at the point \((a, b)\) are denoted by

\[
\left. \frac{\partial z}{\partial x} \right|_{(a, b)} = f_x(a, b) \quad \text{and} \quad \left. \frac{\partial z}{\partial y} \right|_{(a, b)} = f_y(a, b).
\]

**Example 2: Finding and Evaluating Partial Derivatives**

For \( f(x, y) = xe^{x+y} \), find \( f_x \) and \( f_y \), and evaluate each at the point \((1, \ln 2)\).

**Solution**

Because

\[ f_x(x, y) = xe^{x+y}(2xy) + e^{x+y} \]

the partial derivative of \( f \) with respect to \( x \) at \((1, \ln 2)\) is

\[
f_x(1, \ln 2) = e^{\ln 2}(2 \ln 2) + e^{\ln 2} = 4 \ln 2 + 2.
\]

Because

\[ f_y(x, y) = xe^{x+y}(x^2) + e^{x+y} \]

the partial derivative of \( f \) with respect to \( y \) at \((1, \ln 2)\) is

\[
f_y(1, \ln 2) = e^{\ln 2} = 2.
\]

**Try It**

The partial derivatives of a function of two variables, \( z = f(x, y) \), have a useful geometric interpretation. If \( y = y_0 \), then \( z = f(x, y_0) \) represents the curve formed by intersecting the surface \( z = f(x, y) \) with the plane \( y = y_0 \), as shown in Figure 13.29. Therefore,

\[
f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}
\]

represents the slope of this curve at the point \((x_0, y_0, f(x_0, y_0))\). Note that both the curve and the tangent line lie in the plane \( y = y_0 \). Similarly,

\[
f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}
\]

represents the slope of the curve given by the intersection of \( z = f(x, y) \) and the plane \( x = x_0 \) at \((x_0, y_0, f(x_0, y_0))\), as shown in Figure 13.30.

Informally, the values of \( \partial f/\partial x \) and \( \partial f/\partial y \) at the point \((x_0, y_0, z_0)\) denote the slopes of the surface in the \( x \)- and \( y \)-directions, respectively.
EXAMPLE 3  Finding the Slopes of a Surface in the x- and y-Directions

Find the slopes in the x-direction and in the y-direction of the surface given by

\[ f(x, y) = \frac{-x^2}{2} - y^2 + \frac{25}{8} \]

at the point \((\frac{1}{2}, 1, 2)\).

Solution  The partial derivatives of \( f \) with respect to \( x \) and \( y \) are

\[ f_x(x, y) = -x \quad \text{and} \quad f_y(x, y) = -2y. \]

Partial derivatives

So, in the x-direction, the slope is

\[ f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2} \]

Figure 13.31(a)

and in the y-direction, the slope is

\[ f_y\left(\frac{1}{2}, 1\right) = -2. \]

Figure 13.31(b)

EXAMPLE 4  Finding the Slopes of a Surface in the x- and y-Directions

Find the slopes of the surface given by

\[ f(x, y) = 1 - (x - 1)^2 - (y - 2)^2 \]

at the point \((1, 2, 1)\) in the x-direction and in the y-direction.

Solution  The partial derivatives of \( f \) with respect to \( x \) and \( y \) are

\[ f_x(x, y) = -2(x - 1) \quad \text{and} \quad f_y(x, y) = -2(y - 2). \]

Partial derivatives

So, at the point \((1, 2, 1)\), the slopes in the x- and y-directions are

\[ f_x(1, 2) = -2(1 - 1) = 0 \quad \text{and} \quad f_y(1, 2) = -2(2 - 2) = 0 \]

as shown in Figure 13.32.
No matter how many variables are involved, partial derivatives can be interpreted as *rates of change*.

**EXAMPLE 5  Using Partial Derivatives to Find Rates of Change**

The area of a parallelogram with adjacent sides $a$ and $b$ and included angle $\theta$ is given by $A = ab \sin \theta$, as shown in Figure 13.33.

a. Find the rate of change of $A$ with respect to $a$ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$

b. Find the rate of change of $A$ with respect to $\theta$ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$

**Solution**

a. To find the rate of change of the area with respect to $a$, hold $b$ and $\theta$ constant and differentiate with respect to $a$ to obtain

$$\frac{\partial A}{\partial a} = b \sin \theta$$

Find partial with respect to $a$.

$$\frac{\partial A}{\partial a} = 20 \sin \frac{\pi}{6} = 10.$$ Substitute for $b$ and $\theta$.

b. To find the rate of change of the area with respect to $\theta$, hold $a$ and $b$ constant and differentiate with respect to $\theta$ to obtain

$$\frac{\partial A}{\partial \theta} = ab \cos \theta$$

Find partial with respect to $\theta$.

$$\frac{\partial A}{\partial \theta} = 200 \cos \frac{\pi}{6} = 100\sqrt{3}.$$ Substitute for $a$, $b$, and $\theta$.

---

**Partial Derivatives of a Function of Three or More Variables**

The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, there are three partial derivatives, each of which is formed by holding two of the variables constant. That is, to define the partial derivative of $w$ with respect to $x$, consider $y$ and $z$ to be constant and differentiate with respect to $x$. A similar process is used to find the derivatives of $w$ with respect to $y$ and with respect to $z$.

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$, there are $n$ partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$ To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.
EXAMPLE 6 Finding Partial Derivatives

a. To find the partial derivative of \( f(x, y, z) = xy + yz^2 + xz \) with respect to \( z \), consider \( x \) and \( y \) to be constant and obtain

\[
\frac{\partial}{\partial z}[xy + yz^2 + xz] = yz + x.
\]

b. To find the partial derivative of \( f(x, y, z) = z \sin(xy^2 + 2z) \) with respect to \( z \), consider \( x \) and \( y \) to be constant. Then, using the Product Rule, you obtain

\[
\frac{\partial}{\partial z}[z \sin(xy^2 + 2z)] = (z)\frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + \sin(xy^2 + 2z)\frac{\partial}{\partial z}[z]
\]

\[
= (z)\cos(xy^2 + 2z)(2y) + \sin(xy^2 + 2z)
\]

\[
= 2z \cos(xy^2 + 2z) + \sin(xy^2 + 2z).
\]

c. To find the partial derivative of \( f(x, y, z, w) = \frac{x + y + z}{w} \) with respect to \( w \), consider \( x \), \( y \), and \( z \) to be constant and obtain

\[
\frac{\partial}{\partial w}\left[ \frac{x + y + z}{w} \right] = -\frac{x + y + z}{w^2}.
\]

Try It

Exploration A

Higher-Order Partial Derivatives

As is true for ordinary derivatives, it is possible to take second, third, and higher partial derivatives of a function of several variables, provided such derivatives exist. Higher-order derivatives are denoted by the order in which the differentiation occurs. For instance, the function \( z = f(x, y) \) has the following second partial derivatives.

1. Differentiate twice with respect to \( x \):

\[
\frac{\partial}{\partial x}\left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.
\]

2. Differentiate twice with respect to \( y \):

\[
\frac{\partial}{\partial y}\left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.
\]

3. Differentiate first with respect to \( x \) and then with respect to \( y \):

\[
\frac{\partial}{\partial y}\left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.
\]

4. Differentiate first with respect to \( y \) and then with respect to \( x \):

\[
\frac{\partial}{\partial x}\left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.
\]

The third and fourth cases are called mixed partial derivatives.
EXAMPLE 7  Finding Second Partial Derivatives

Find the second partial derivatives of \( f(x, y) = 3xy^2 - 2y + 5x^2y^2 \), and determine the value of \( f_{xy}(-1, 2) \).

Solution  Begin by finding the first partial derivatives with respect to \( x \) and \( y \).
\[
\begin{align*}
  f_x(x, y) &= 3y^2 + 10xy^2 & \quad & f_y(x, y) &= 6xy - 2 + 10x^2y
\end{align*}
\]
Then, differentiate each of these with respect to \( x \) and \( y \).
\[
\begin{align*}
  f_{xx}(x, y) &= 10y^2 & \quad & f_{yx}(x, y) &= 6x + 10x^2 \\
  f_{xy}(x, y) &= 6y + 20xy & \quad & f_{yy}(x, y) &= 6y + 20xy
\end{align*}
\]
At \((-1, 2)\), the value of \( f_{xy} \) is \( f_{xy}(-1, 2) = 12 - 40 = -28 \).

NOTE  Notice in Example 7 that the two mixed partials are equal. Sufficient conditions for this occurrence are given in Theorem 13.3.

THEOREM 13.3  Equality of Mixed Partial Derivatives

If \( f \) is a function of \( x \) and \( y \) such that \( f_{xy} \) and \( f_{yx} \) are continuous on an open disk \( R \), then, for every \((x, y)\) in \( R \),
\[
f_{xy}(x, y) = f_{yx}(x, y).
\]

Theorem 13.3 also applies to a function \( f \) of \( \text{three or more variables} \) so long as all second partial derivatives are continuous. For example, if \( w = f(x, y, z) \) and all the second partial derivatives are continuous in an open region \( R \), then at each point in \( R \) the order of differentiation in the mixed second partial derivatives is irrelevant. If the third partial derivatives of \( f \) are also continuous, the order of differentiation of the mixed third partial derivatives is irrelevant.

EXAMPLE 8  Finding Higher-Order Partial Derivatives

Show that \( f_{xz} = f_{zx} \) and \( f_{zxx} = f_{xxz} = f_{xxz} \) for the function given by
\[
f(x, y, z) = ye^x + x \ln z.
\]

Solution

First partials:
\[
\begin{align*}
  f_x(x, y, z) &= ye^x + \ln z, & \quad & f_y(x, y, z) &= \frac{x}{z}
\end{align*}
\]
Second partials (note that the first two are equal):
\[
\begin{align*}
  f_{xx}(x, y, z) &= \frac{1}{z}, & \quad & f_{yx}(x, y, z) &= \frac{1}{z}, & \quad & f_{xy}(x, y, z) &= -\frac{x}{z^2}
\end{align*}
\]
Third partials (note that all three are equal):
\[
\begin{align*}
  f_{xxx}(x, y, z) &= -\frac{1}{z^2}, & \quad & f_{zx}(x, y, z) &= -\frac{1}{z^2}, & \quad & f_{xzx}(x, y, z) &= -\frac{1}{z^2}
\end{align*}
\]
Exercises for Section 13.3

The symbol \( \textcolor{red}{\text{H}} \) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on \( \textcolor{blue}{\text{S}} \) to view the complete solution of the exercise.

Click on \( \textcolor{magenta}{\text{M}} \) to print an enlarged copy of the graph.

Think About It  In Exercises 1–4, use the graph of the surface to determine the sign of the indicated partial derivative.

In Exercises 5–28, find both first partial derivatives.

5. \( f(x, y) = 2x - 3y + 5 \)
6. \( f(x, y) = x^2 - 3y^2 + 7 \)
7. \( z = x\sqrt{y} \)
8. \( z = 2y^2 \sqrt{x} \)
9. \( z = x^2 - 5xy + 3y^2 \)
10. \( z = y^3 - 4xy^2 - 1 \)
11. \( z = xe^{xy} \)
12. \( z = x^e^{xy} \)
13. \( z = \ln(x^2 + y^2) \)
14. \( z = \ln \sqrt{xy} \)
15. \( z = \ln \frac{x + y}{x - y} \)
16. \( z = \ln(x^2 - y^2) \)
17. \( z = \frac{x^2 + 4y^2}{2y} \)
18. \( z = \frac{xy}{x^2 + y^2} \)
19. \( h(x, y) = e^{-(x^2+y^2)} \)
20. \( g(x, y) = \ln \sqrt{x^2 + y^2} \)
21. \( f(x, y) = \sqrt{x^2 + y^2} \)
22. \( f(x, y) = \sqrt{2x^2 + y^2} \)
23. \( z = \tan(2x - y) \)
24. \( z = \sin 3x \cos 3y \)
25. \( z = e^x \sin xy \)
26. \( z = \cos(x^2 + y^2) \)
27. \( f(x, y) = \int_1^x (t^2 - 1) \, dt \)
28. \( f(x, y) = \int_1^y (2t + 1) \, dt + \int_1^y (2t - 1) \, dt \)

In Exercises 29–32, use the limit definition of partial derivatives to find \( f_x(x, y) \) and \( f_y(x, y) \).

29. \( f(x, y) = 2x + 3y \)
30. \( f(x, y) = x^2 - 2xy + y^2 \)
31. \( f(x, y) = \sqrt{x + y} \)
32. \( f(x, y) = \frac{1}{x + y} \)

In Exercises 33–36, evaluate \( f_x \) and \( f_y \) at the given point.

33. \( f(x, y) = \arctan \frac{y}{x} \) \( (2, -2) \)
34. \( f(x, y) = \arccos xy \) \( (1, 1) \)
35. \( f(x, y) = \frac{xy}{x - y} \) \( (2, -2) \)
36. \( f(x, y) = \frac{6xy}{\sqrt{x^2 + y^2}} \) \( (1, 1) \)

In Exercises 37–40, find the slopes of the surface in the \( x \)- and \( y \)-directions at the given point.

37. \( g(x, y) = 4 - x^2 - y^2 \) \( (1, 1, 2) \)
38. \( h(x, y) = x^2 - y^2 \) \( (-2, 1, 3) \)

In Exercises 41–44, use a computer algebra system to graph the curve formed by the intersection of the surface and the plane. Find the slope of the curve at the given point.

<table>
<thead>
<tr>
<th>Surface</th>
<th>Plane</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = \sqrt{49 - x^2 - y^2} )</td>
<td>( x = 2 )</td>
<td>( (2, 3, 6) )</td>
</tr>
<tr>
<td>( z = x^2 + 4y^2 )</td>
<td>( y = 1 )</td>
<td>( (2, 1, 8) )</td>
</tr>
<tr>
<td>( z = 9x^2 - y^2 )</td>
<td>( y = 3 )</td>
<td>( (3, 1, 0) )</td>
</tr>
<tr>
<td>( z = 9x^2 - y^2 )</td>
<td>( x = 1 )</td>
<td>( (3, 1, 0) )</td>
</tr>
</tbody>
</table>

In Exercises 45–48, for \( f(x, y) \), find all values of \( x \) and \( y \) such that \( f_x(x, y) = 0 \) and \( f_y(x, y) = 0 \) simultaneously.

45. \( f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3 \)
46. \( f(x, y) = 3x^2 - 12xy + y^3 \)
47. \( f(x, y) = \frac{1}{x} + \frac{1}{y} + xy \)
48. \( f(x, y) = \ln(x^2 + y^2 + 1) \)
Think About It  In Exercises 49 and 50, the graph of a function \( f \) and its two partial derivatives \( f_x \) and \( f_y \) are given. Identify \( f_x \) and \( f_y \) and give reasons for your answers.

49.

50.

In Exercises 51–56, find the first partial derivatives with respect to \( x \), \( y \), and \( z \), and give reasons for your answers.

51. \( w = \sqrt{x^2 + y^2 + z^2} \)
52. \( w = \frac{3xy}{x + y} \)
53. \( F(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2} \)
54. \( G(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}} \)
55. \( H(x, y, z) = \sin(x + 2y + 3z) \)
56. \( f(x, y, z) = 3x^2y - 5xyz + 10yz^2 \)

In Exercises 57–60, evaluate \( f_x \), \( f_y \), and \( f_z \) at the given point.

57. \( f(x, y, z) = \sqrt{3x^2 + y^2 - 2z^2}, \quad (1, -2, 1) \)
58. \( f(x, y, z) = \frac{xy}{x + y + z}, \quad (3, 1, -1) \)

In Exercises 57–60, evaluate \( f_x \), \( f_y \), and \( f_z \) at the given point.

59. \( f(x, y, z) = z \sin(x + y), \quad \left(0, \frac{\pi}{2}, -4\right) \)
60. \( f(x, y, z) = x^3y^3 + 2xyz - 3yz, \quad (-2, 1, 2) \)

In Exercises 61–68, find the four second partial derivatives. Observe that the second mixed partials are equal.

61. \( z = x^2 - 2xy + 3y^2 \)
62. \( z = x^4 - 3x^2y^2 + y^4 \)
63. \( z = \sqrt{x^2 + y^2} \)
64. \( z = \ln(x - y) \)
65. \( z = e^x \tan y \)
66. \( z = 2xe^y - 3ye^{-x} \)
67. \( z = \arctan \frac{y}{x} \)
68. \( z = \sin(x - 2y) \)

In Exercises 69–72, use a computer algebra system to find the first and second partial derivatives of the function. Determine whether there exist values of \( x \) and \( y \) such that \( f_x(x, y) = 0 \) and \( f_y(x, y) = 0 \) simultaneously.

69. \( f(x, y) = x \sec y \)
70. \( f(x, y) = \sqrt{9 - x^2 - y^2} \)
71. \( f(x, y) = \ln \frac{x}{x^2 + y^2} \)
72. \( f(x, y) = \frac{xy}{x - y} \)

In Exercises 73–76, show that the mixed partial derivatives \( f_{xy} \), \( f_{xy} \), and \( f_{yx} \) are equal.

73. \( f(x, y, z) = xyz \)
74. \( f(x, y, z) = x^2 - 3xy + 4yz + z^3 \)
75. \( f(x, y, z) = e^{-z} \sin yz \)
76. \( f(x, y, z) = \frac{2z}{x + y} \)

Laplace’s Equation  In Exercises 77–80, show that the function satisfies Laplace’s equation \( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \).

77. \( z = 5xy \)
78. \( z = \frac{1}{2}(e^x - e^{-x}) \sin x \)
79. \( z = e^x \sin y \)
80. \( z = \arctan \frac{y}{x} \)

Wave Equation  In Exercises 81–84, show that the function satisfies the wave equation \( \frac{\partial^2 z}{\partial t^2} = c^2 \left( \frac{\partial^2 z}{\partial x^2} \right) \).

81. \( z = \sin(x - ct) \)
82. \( z = \cos(4x + 4ct) \)
83. \( z = \ln(x + ct) \)
84. \( z = \sin \omega x \sin \omega ct \)

Heat Equation  In Exercises 85 and 86, show that the function satisfies the heat equation \( \frac{\partial z}{\partial t} = c^2 \left( \frac{\partial^2 z}{\partial x^2} \right) \).

85. \( z = e^{-t} \cos \frac{x}{c} \)
86. \( z = e^{-t} \sin \frac{x}{c} \)
98. **Apparent Temperature**  A measure of what hot weather feels like to two average persons is the Apparent Temperature Index. A model for this index is

\[ A = 0.885t - 22.4h + 1.20th - 0.544 \]

where \( A \) is the apparent temperature in degrees Celsius, \( t \) is the air temperature, and \( h \) is the relative humidity in decimal form. (Source: The UMAP Journal, Fall 1984)

(a) Find \( \partial A/\partial t \) and \( \partial A/\partial h \) when \( t = 30^\circ \) and \( h = 0.80 \).
(b) Which has a greater effect on \( A \), air temperature or humidity? Explain.

99. **Ideal Gas Law**  The Ideal Gas Law states that \( PV = nRT \) where \( P \) is pressure, \( V \) is volume, \( n \) is the number of moles of gas, \( R \) is a fixed constant (the gas constant), and \( T \) is absolute temperature. Show that

\[ \frac{\partial T}{\partial P} \frac{\partial P}{\partial V} \frac{\partial V}{\partial T} = -1. \]

100. **Marginal Utility**  The utility function \( U = f(x, y) \) is a measure of the utility (or satisfaction) derived by a person from the consumption of two products \( x \) and \( y \). Suppose the utility function is

\[ U = -5x^2 + xy - 3y^2. \]

(a) Determine the marginal utility of product \( x \).
(b) Determine the marginal utility of product \( y \).
(c) When \( x = 2 \) and \( y = 3 \), should a person consume one more unit of product \( x \) or one more unit of product \( y \)? Explain your reasoning.
(d) Use a computer algebra system to graph the function

Interpret the marginal utilities of products \( x \) and \( y \) graphically.

101. **Modeling Data**  Per capita consumptions (in gallons) of different types of plain milk in the United States from 1994 to 2000 are shown in the table. Consumption of light and skim milks, reduced-fat milk, and whole milk are represented by the variables \( x \), \( y \), and \( z \), respectively. (Source: U.S. Department of Agriculture)

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>( x )</td>
<td>5.8</td>
<td>6.2</td>
<td>6.4</td>
<td>6.6</td>
<td>6.5</td>
<td>6.3</td>
<td>6.1</td>
</tr>
<tr>
<td>( y )</td>
<td>8.7</td>
<td>8.2</td>
<td>8.0</td>
<td>7.7</td>
<td>7.4</td>
<td>7.3</td>
<td>7.1</td>
</tr>
<tr>
<td>( z )</td>
<td>8.8</td>
<td>8.4</td>
<td>8.4</td>
<td>8.2</td>
<td>7.8</td>
<td>7.9</td>
<td>7.8</td>
</tr>
</tbody>
</table>

A model for the data is given by

\[ z = -0.04x + 0.64y + 3.4. \]

(a) Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).
(b) Interpret the partial derivatives in the context of the problem.
102. **Modeling Data** The table shows the amount of public medical expenditures (in billions of dollars) for worker’s compensation \( x \), public assistance \( y \), and Medicare \( z \) for selected years. \((Source: Centers for Medicare and Medicaid Services)\)

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>17.5</td>
<td>21.9</td>
<td>20.5</td>
<td>20.8</td>
<td>22.5</td>
<td>23.3</td>
</tr>
<tr>
<td>( y )</td>
<td>78.7</td>
<td>157.6</td>
<td>164.8</td>
<td>176.6</td>
<td>191.8</td>
<td>208.5</td>
</tr>
<tr>
<td>( z )</td>
<td>110.2</td>
<td>197.5</td>
<td>208.2</td>
<td>209.5</td>
<td>212.6</td>
<td>224.4</td>
</tr>
</tbody>
</table>

A model for the data is given by

\[
z = -1.3520x^2 - 0.0025y^2 + 56.080x + 1.537y - 562.23.
\]

(a) Find \( \frac{\partial^2 z}{\partial x^2} \) and \( \frac{\partial^2 z}{\partial y^2} \).

(b) Determine the concavity of traces parallel to the \( xz \)-plane.

Interpret the result in the context of the problem.

(c) Determine the concavity of traces parallel to the \( yz \)-plane.

Interpret the result in the context of the problem.

**True or False?** In Exercises 103–106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

103. If \( z = f(x, y) \) and \( \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \), then \( z = c(x + y) \).

104. If \( z = f(x)g(y) \), then \( \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f'(x)g(y) + f(x)g'(y) \).

105. If \( z = e^y \), then \( \frac{\partial^2 z}{\partial y \partial x} = (xy + 1)e^y \).

106. If a cylindrical surface \( z = f(x, y) \) has rulings parallel to the \( y \)-axis, then \( \frac{\partial z}{\partial y} = 0 \).

107. Consider the function defined by

\[
f(x, y) = \begin{cases} 
xy(x^2 - y^2), & (x, y) \neq (0, 0) \\
\frac{x^2}{x^2 + y^2}, & (x, y) = (0, 0) 
\end{cases}
\]

(a) Find \( f_x(x, y) \) and \( f_y(x, y) \) for \( (x, y) \neq (0, 0) \).

(b) Use the definition of partial derivatives to find \( f_x(0, 0) \) and \( f_y(0, 0) \).

\[
\text{Hint: } f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}
\]

(c) Use the definition of partial derivatives to find \( f_{yx}(0, 0) \) and \( f_{xy}(0, 0) \).

(d) Using Theorem 13.3 and the result of part (c), what can be said about \( f_{xx} \) or \( f_{yy} \)?

108. Let \( f(x, y) = \int_x^y \sqrt{1 + t^2} \, dt \). Find \( f_x(x, y) \) and \( f_y(x, y) \).

109. Consider the function \( f(x, y) = (x^3 + y^3)^{1/3} \).

(a) Show that \( f_x(0, 0) = 1 \).

(b) Determine the points (if any) at which \( f_x(x, y) \) fails to exist.

110. Consider the function \( f(x, y) = (x^2 + y^2)^{2/3} \). Show that

\[
f_x(x, y) = \begin{cases} 
\frac{4x}{3(x^2 + y^2)^{1/3}}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0) 
\end{cases}
\]

**FOR FURTHER INFORMATION** For more information about this problem, see the article “A Classroom Note on a Naturally Occurring Piecewise Defined Function” by Don Cohen.
Differentials

- Understand the concepts of increments and differentials.
- Extend the concept of differentiability to a function of two variables.
- Use a differential as an approximation.

Increments and Differentials

In this section, the concepts of increments and differentials are generalized to functions of two or more variables. Recall from Section 3.9 that for \( y = f(x) \), the differential of \( y \) was defined as

\[
dy = f'(x) \, dx.
\]

Similar terminology is used for a function of two variables, \( z = f(x, y) \). That is, \( \Delta x \) and \( \Delta y \) are the increments of \( x \) and \( y \), and the increment of \( z \) is given by

\[
\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).
\]

Definition of Total Differential

If \( z = f(x, y) \) and \( \Delta x \) and \( \Delta y \) are increments of \( x \) and \( y \), then the differentials of the independent variables \( x \) and \( y \) are

\[
dx = \Delta x \quad \text{and} \quad dy = \Delta y
\]

and the total differential of the dependent variable \( z \) is

\[
dz = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy = f_x(x, y) \, dx + f_y(x, y) \, dy.
\]

This definition can be extended to a function of three or more variables. For instance, if \( w = f(x, y, z, u) \), then \( dx = \Delta x, dy = \Delta y, dz = \Delta z, du = \Delta u \), and the total differential of \( w \) is

\[
dw = \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy + \frac{\partial w}{\partial z} \, dz + \frac{\partial w}{\partial u} \, du.
\]

EXAMPLE 1  Finding the Total Differential

Find the total differential for each function.

a. \( z = 2x \sin y - 3x^2y^2 \)  \hspace{1cm} b. \( w = x^2 + y^2 + z^2 \)

Solution

a. The total differential \( dz \) for \( z = 2x \sin y - 3x^2y^2 \) is

\[
dz = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy = (2 \sin y - 6xy^2) \, dx + (2x \cos y - 6x^2y) \, dy.
\]

b. The total differential \( dw \) for \( w = x^2 + y^2 + z^2 \) is

\[
dw = \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy + \frac{\partial w}{\partial z} \, dz = 2x \, dx + 2y \, dy + 2z \, dz.
\]

Try It  Exploration A
**Differentiability**

In Section 3.9, you learned that for a *differentiable* function given by \( y = f(x) \), you can use the differential \( dy = f'(x) \, dx \) as an approximation (for small \( \Delta x \)) to the value \( \Delta y = f(x + \Delta x) - f(x) \). When a similar approximation is possible for a function of two variables, the function is said to be *differentiable*. This is stated explicitly in the following definition.

**Definition of Differentiability**

A function \( f \) given by \( z = f(x, y) \) is *differentiable* at \((x_0, y_0)\) if \( \Delta z \) can be written in the form

\[
\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\]

where both \( \varepsilon_1 \) and \( \varepsilon_2 \to 0 \) as \((\Delta x, \Delta y) \to (0, 0)\). The function \( f \) is differentiable in a region \( R \) if it is differentiable at each point in \( R \).

**EXAMPLE 2**  **Showing That a Function Is Differentiable**

Show that the function given by

\[ f(x, y) = x^2 + 3y \]

is differentiable at every point in the plane.

**Solution**  Letting \( z = f(x, y) \), the increment of \( z \) at an arbitrary point \((x, y)\) in the plane is

\[
\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \quad \text{Increment of } z
\]

\[
= (x^2 + 2x\Delta x + \Delta x^2) + 3(y + \Delta y) - (x^2 + 3y)
\]

\[
= 2x\Delta x + \Delta x^2 + 3\Delta y
\]

\[
= 2x(\Delta x) + 3(\Delta y) + \Delta x(\Delta y) + 0(\Delta y)
\]

\[
= f_x(x, y) \Delta x + f_y(x, y) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\]

where \( \varepsilon_1 = \Delta x \) and \( \varepsilon_2 = 0 \). Because \( \varepsilon_1 \to 0 \) and \( \varepsilon_2 \to 0 \) as \((\Delta x, \Delta y) \to (0, 0)\), it follows that \( f \) is differentiable at every point in the plane. The graph of \( f \) is shown in Figure 13.34.

**Try It**  **Exploration A**

Be sure you see that the term “differentiable” is used differently for functions of two variables than for functions of one variable. A function of one variable is differentiable at a point if its derivative exists at the point. However, for a function of two variables, the existence of the partial derivatives \( f_x \) and \( f_y \) does not guarantee that the function is differentiable (see Example 5). The following theorem gives a sufficient condition for differentiability of a function of two variables. A proof of Theorem 13.4 is given in Appendix A.

**THEOREM 13.4**  **Sufficient Condition for Differentiability**

If \( f \) is a function of \( x \) and \( y \), where \( f_x \) and \( f_y \) are continuous in an open region \( R \), then \( f \) is differentiable on \( R \).
Approximation by Differentials

Theorem 13.4 tells you that you can choose \((x + \Delta x, y + \Delta y)\) close enough to \((x, y)\) to make \(\varepsilon_x \Delta x\) and \(\varepsilon_y \Delta y\) insignificant. In other words, for small \(\Delta x\) and \(\Delta y\), you can use the approximation

\[
\Delta z \approx dz.
\]

This approximation is illustrated graphically in Figure 13.35. Recall that the partial derivatives \(\partial z/\partial x\) and \(\partial z/\partial y\) can be interpreted as the slopes of the surface in the \(x\)- and \(y\)-directions. This means that

\[
dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y
\]

represents the change in height of a plane that is tangent to the surface at the point \((x, y, f(x, y))\). Because a plane in space is represented by a linear equation in the variables \(x\), \(y\), and \(z\), the approximation of \(\Delta z\) by \(dz\) is called a linear approximation. You will learn more about this geometric interpretation in Section 13.7.

**EXAMPLE 3 Using a Differential as an Approximation**

Use the differential \(dz\) to approximate the change in \(z = \sqrt{4 - x^2 - y^2}\) as \((x, y)\) moves from the point \((1, 1)\) to the point \((1.01, 0.97)\). Compare this approximation with the exact change in \(z\).

**Solution** Letting \((x, y) = (1, 1)\) and \((x + \Delta x, y + \Delta y) = (1.01, 0.97)\) produces \(dx = \Delta x = 0.01\) and \(dy = \Delta y = -0.03\). So, the change in \(z\) can be approximated by

\[
\Delta z \approx dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{-x}{\sqrt{4 - x^2 - y^2}} \Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}} \Delta y.
\]

When \(x = 1\) and \(y = 1\), you have

\[
\Delta z \approx -\frac{1}{\sqrt{2}} (0.01) - \frac{1}{\sqrt{2}} (-0.03) = 0.02 = \sqrt{2} (0.01) \approx 0.0141.
\]

In Figure 13.36 you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere. This difference is given by

\[
\Delta z = f(1.01, 0.97) - f(1, 1) = \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2} = 0.0137.
\]

**Try It**

A function of three variables \(w = f(x, y, z)\) is called differentiable at \((x, y, z)\) provided that

\[
\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)
\]

can be written in the form

\[
\Delta w = f_x \Delta x + f_y \Delta y + f_z \Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z
\]

where \(\varepsilon_1, \varepsilon_2,\) and \(\varepsilon_3 \to 0\) as \((\Delta x, \Delta y, \Delta z) \to (0, 0, 0)\). With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables: If \(f\) is a function of \(x, y,\) and \(z\), where \(f, f_x, f_y,\) and \(f_z\) are continuous in an open region \(R\), then \(f\) is differentiable on \(R\).

In Section 3.9, you used differentials to approximate the propagated error introduced by an error in measurement. This application of differentials is further illustrated in Example 4.
**EXAMPLE 4  Error Analysis**

The possible error involved in measuring each dimension of a rectangular box is ±0.1 millimeter. The dimensions of the box are \( x = 50 \) centimeters, \( y = 20 \) centimeters, and \( z = 15 \) centimeters, as shown in Figure 13.37. Use \( dV \) to estimate the propagated error and the relative error in the calculated volume of the box.

**Solution**

The volume of the box is given by \( V = xyz \), and so

\[
dV = \frac{\partial V}{\partial x} \, dx + \frac{\partial V}{\partial y} \, dy + \frac{\partial V}{\partial z} \, dz
\]

\[= yz \, dx + xz \, dy + xy \, dz.\]

Using 0.1 millimeter = 0.01 centimeter, you have \( dx = dy = dz = ±0.01 \), and the propagated error is approximately

\[
dV = (20)(15)(±0.01) + (50)(15)(±0.01) + (50)(20)(±0.01)
\]

\[= 300(±0.01) + 750(±0.01) + 1000(±0.01)
\]

\[= 2050(±0.01) = ±20.5 \text{ cubic centimeters}.\]

Because the measured volume is

\[V = (50)(20)(15) = 15,000 \text{ cubic centimeters},\]

the relative error, \( \Delta V/V \), is approximately

\[
\Delta V/V \approx \frac{dV}{V} = \frac{20.5}{15,000} \approx 0.14%.
\]

**Try It**  
As is true for a function of a single variable, if a function in two or more variables is differentiable at a point, it is also continuous there.

**THEOREM 13.5  Differentiability Implies Continuity**

If a function \( f \) of \( x \) and \( y \) is differentiable at \((x_0, y_0)\), then it is continuous at \((x_0, y_0)\).

**Proof**  
Let \( f \) be differentiable at \((x_0, y_0)\), where \( z = f(x, y) \). Then

\[
\Delta z = [f(x_0, y_0) + e_1] \Delta x + [f(x_0, y_0) + e_2] \Delta y
\]

where both \( e_1 \) and \( e_2 \to 0 \) as \((\Delta x, \Delta y) \to (0, 0)\). However, by definition, you know that \( \Delta z \) is given by

\[
\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).
\]

Letting \( x = x_0 + \Delta x \) and \( y = y_0 + \Delta y \) produces

\[
f(x, y) - f(x_0, y_0) = [f(x_0, y_0) + e_1] \Delta x + [f(x_0, y_0) + e_2] \Delta y
\]

\[= [f(x_0, y_0) + e_1](x - x_0) + [f(x_0, y_0) + e_2](y - y_0).
\]

Taking the limit as \((x, y) \to (x_0, y_0)\), you have

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)
\]

which means that \( f \) is continuous at \((x_0, y_0)\).
Remember that the existence of $f_x$ and $f_y$ is not sufficient to guarantee differentiability, as illustrated in the next example.

**EXAMPLE 5 A Function That Is Not Differentiable**

Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that $f$ is not differentiable at $(0, 0)$ where $f$ is defined as

$$f(x, y) = \begin{cases} \frac{-3xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

**Solution** You can show that $f$ is not differentiable at $(0, 0)$ by showing that it is not continuous at this point. To see that $f$ is not continuous at $(0, 0)$, look at the values of $f(x, y)$ along two different approaches to $(0, 0)$, as shown in Figure 13.38. Along the line $y = x$, the limit is

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, x) \to (0, 0)} \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

whereas along $y = -x$ you have

$$\lim_{(x, -x) \to (0, 0)} f(x, y) = \lim_{(x, -x) \to (0, 0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$  

So, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist, and you can conclude that $f$ is not continuous at $(0, 0)$. Therefore, by Theorem 13.5, you know that $f$ is not differentiable at $(0, 0)$. On the other hand, by the definition of the partial derivatives $f_x$ and $f_y$, you have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

and

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0.$$  

So, the partial derivatives at $(0, 0)$ exist.

$$f(x, y) = \begin{cases} \frac{-3xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Along the line $y = -x$, $f(x, y)$ approaches $3/2$.  
Along the line $y = x$, $f(x, y)$ approaches $-3/2$.  

**Figure 13.38**
The symbol + indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on to view the complete solution of the exercise.

Click on to print an enlarged copy of the graph.

In Exercises 1–10, find the total differential.

1. \( z = 3x^2y^3 \)
2. \( z = \frac{x^2}{y} \)
3. \( z = \frac{-1}{x^2 + y^2} \)
4. \( w = x + y \frac{z}{2y} \)
5. \( z = x \cos y - y \cos x \)
6. \( z = \frac{1}{2}(e^x + y^2 - e^{-x} - y^2) \)
7. \( z = e^x \sin y \)
8. \( w = e^x \cos x + z^2 \)
9. \( w = 2e^x \sin x \)
10. \( w = x^3y^2 + \sin yz \)

In Exercises 11–16, (a) evaluate \( f(1, 2) \) and \( f(1.05, 2.1) \) and calculate \( \Delta z \), and (b) use the total differential \( dz \) to approximate \( \Delta z \).

11. \( f(x, y) = 9 - x^2 - y^2 \)
12. \( f(x, y) = \sqrt{x^2 + y^2} \)
13. \( f(x, y) = x \sin y \)
14. \( f(x, y) = xe^y \)
15. \( f(x, y) = 3x - 4y \)
16. \( f(x, y) = \frac{x}{y} \)

In Exercises 17–20, find \( z = f(x, y) \) and use the total differential to approximate the quantity.

17. \( \sqrt{(5.05)^2 - (3.1)^2} - \sqrt{3^2 + 3^2} \)
18. \( (2.03)^2(1 + 8.9)^3 - 2(1 + 9)^3 \)
19. \( \frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \)
20. \( \sin[(1.05)^2 + (0.95)^2] - \sin(1^2 + 1^2) \)

Writing About Concepts

21. Define the total differential of a function of two variables.
22. Describe the change in accuracy of \( dz \) as an approximation of \( \Delta z \) as \( \Delta x \) and \( \Delta y \) increase.
23. What is meant by a linear approximation of \( z = f(x, y) \) at the point \( P(x_0, y_0) \)?
24. When using differentials, what is meant by the terms propagated error and relative error?

25. Area The area of the shaded rectangle in the figure is \( A = lh \). The possible errors in the length and height are \( \Delta l \) and \( \Delta h \), respectively. Find \( dA \) and identify the regions in the figure whose areas are given by the terms of \( dA \). What region represents the difference between \( \Delta A \) and \( dA \)?

26. Volume The volume of the red right circular cylinder in the figure is \( V = \pi r^2 h \). The possible errors in the radius and the height are \( \Delta r \) and \( \Delta h \), respectively. Find \( dV \) and identify the solids in the figure whose volumes are given by the terms of \( dV \). What solid represents the difference between \( \Delta V \) and \( dV \)?

27. Numerical Analysis A right circular cone of height \( h = 6 \) and radius \( r = 3 \) is constructed, and in the process errors \( \Delta r \) and \( \Delta h \) are made in the radius and height, respectively. Complete the table to show the relationship between \( \Delta V \) and \( dV \) for the indicated errors.

<table>
<thead>
<tr>
<th>( \Delta r )</th>
<th>( \Delta h )</th>
<th>( dV ) or ( dS )</th>
<th>( \Delta V ) or ( \Delta S )</th>
<th>( \Delta V - dV ) or ( \Delta S - dS )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>-0.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.001</td>
<td>0.002</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.0001</td>
<td>0.0002</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

28. Numerical Analysis The height and radius of a right circular cone are measured as \( h = 20 \) meters and \( r = 8 \) meters. In the process of measuring, errors \( \Delta r \) and \( \Delta h \) are made. \( S \) is the lateral surface area of a cone. Complete the table above to show the relationship between \( \Delta S \) and \( dS \) for the indicated errors.

29. Modeling Data Per capita consumptions (in gallons) of different types of plain milk in the United States from 1994 to 2000 are shown in the table. Consumption of light and skim milks, reduced-fat milk, and whole milk are represented by the variables \( x, y, \) and \( z \), respectively. (Source: U.S. Department of Agriculture)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>5.8</td>
<td>6.2</td>
<td>6.4</td>
<td>6.6</td>
<td>6.5</td>
<td>6.3</td>
<td>6.1</td>
</tr>
<tr>
<td>( y )</td>
<td>8.7</td>
<td>8.2</td>
<td>8.0</td>
<td>7.7</td>
<td>7.4</td>
<td>7.3</td>
<td>7.1</td>
</tr>
<tr>
<td>( z )</td>
<td>8.8</td>
<td>8.4</td>
<td>8.4</td>
<td>8.2</td>
<td>7.8</td>
<td>7.9</td>
<td>7.8</td>
</tr>
</tbody>
</table>

A model for the data is given by \( z = -0.04x + 0.64y + 3.4 \).

(a) Find the total differential of the model.

(b) A dairy industry forecast for a future year is that per capita consumption of light and skim milks will be \( 6.2 \pm 0.2 \) gallons and that per capita consumption of reduced-fat milk will be \( 7.5 \pm 0.25 \) gallons. Use \( dz \) to estimate the maximum possible propagated error and relative error in the prediction for the consumption of whole milk.

30. Rectangular to Polar Coordinates A rectangular coordinate system is placed over a map and the coordinates of a point of interest are \((8.5, 3.2)\). There is a possible error of 0.05 in each coordinate. Approximate the maximum possible error in measuring the polar coordinates for the point.
31. **Volume** The radius \( r \) and height \( h \) of a right circular cylinder are measured with possible errors of 4\% and 2\%, respectively. Approximate the maximum possible percent error in measuring the volume.

32. **Area** A triangle is measured and two adjacent sides are found to be 3 inches and 4 inches long, with an included angle of \( \pi/4 \). The possible errors in measurement are \( \pm 3\% \) for the sides and 0.02 radian for the angle. Approximate the maximum possible error in the computation of the area.

33. **Wind Chill** The formula for wind chill \( C \) (in degrees Fahrenheit) is given by

\[
C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275T^{0.16}
\]

where \( v \) is the wind speed in miles per hour and \( T \) is the temperature in degrees Fahrenheit. The wind speed is 23 \( \pm \) 3 miles per hour and the temperature is 8\( ^\circ \) \( \pm \) 1\( ^\circ \). Use \( dC \) to estimate the maximum possible propagated error and relative error in calculating the wind chill.

34. **Acceleration** The centripetal acceleration of a particle moving in a circle is \( a = \frac{v^2}{r} \), where \( v \) is the velocity and \( r \) is the radius of the circle. Approximate the maximum percent error in measuring the acceleration due to errors of 3\% in \( v \) and 2\% in \( r \).

35. **Volume** A trough is 16 feet long (see figure). Its cross sections are isosceles triangles with each of the two equal sides measuring 18 inches. The angle between the two equal sides is \( \theta \). (a) Write the volume of the trough as a function of \( \theta \) and determine the value of \( \theta \) such that the volume is a maximum. (b) The maximum error in the linear measurements is one-half inch and the maximum error in the angle measure is 2\( ^\circ \). Approximate the change from the maximum volume.

![Figure for 35](image)

36. **Sports** A baseball player in center field is playing approximately 330 feet from a television camera that is behind home plate. A batter hits a fly ball that goes to a wall 420 feet from the camera (see figure).

(a) The camera turns 9\( ^\circ \) to follow the play. Approximate the number of feet that the center fielder has to run to make the catch.

(b) The position of the center fielder could be in error by as much as 6 feet and the maximum error in measuring the rotation of the camera is 1\( ^\circ \). Approximate the maximum possible error in the result of part (a).

![Figure for 36](image)

37. **Power** Electrical power \( P \) is given by

\[
P = \frac{E^2}{R}
\]

where \( E \) is voltage and \( R \) is resistance. Approximate the maximum percent error in calculating power if 200 volts is applied to a 4000-ohm resistor and the possible percent errors in measuring \( E \) and \( R \) are 2\% and 3\%, respectively.

38. **Resistance** The total resistance \( R \) of two resistors connected in parallel is

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}
\]

Approximate the change in \( R \) as \( R_1 \) is increased from 10 ohms to 10.5 ohms and \( R_2 \) is decreased from 15 ohms to 13 ohms.

39. **Inductance** The inductance \( L \) (in microhenrys) of a straight nonmagnetic wire in free space is

\[
L = 0.00021 \left( \ln \frac{2h}{r} - 0.75 \right)
\]

where \( h \) is the length of the wire in millimeters and \( r \) is the radius of a circular cross section. Approximate \( L \) when \( r = 2 \pm \frac{1}{10} \) millimeters and \( h = 100 \pm \frac{1}{100} \) millimeters.

40. **Pendulum** The period \( T \) of a pendulum of length \( l \) is

\[
T = 2\pi\sqrt{\frac{l}{g}}
\]

where \( g \) is the acceleration due to gravity. A pendulum is moved from the Canal Zone, where \( g = 32.09 \) feet per second per second, to Greenland, where \( g = 32.23 \) feet per second per second. Because of the change in temperature, the length of the pendulum changes from 2.5 feet to 2.48 feet. Approximate the change in the period of the pendulum.

In Exercises 41–44, show that the function is differentiable by finding values for \( \varepsilon_1 \) and \( \varepsilon_2 \) as designated in the definition of differentiability, and verify that both \( \varepsilon_1 \) and \( \varepsilon_2 \) \( \to 0 \) as \( (\Delta x, \Delta y) \to (0, 0) \).

41. \( f(x, y) = x^2 - 2x + y \)  
42. \( f(x, y) = x^2 + y^2 \)
43. \( f(x, y) = x^2y \)  
44. \( f(x, y) = 5x - 10y + y^3 \)

In Exercises 45 and 46, use the function to prove that (a) \( f_x(0,0) \) and \( f_y(0,0) \) exist, and (b) \( f \) is not differentiable at \( (0, 0) \).

45. \( f(x,y) = \begin{cases} 
\frac{3x^2y^2}{x^4 + y^2}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0)
\end{cases} \)

46. \( f(x,y) = \begin{cases} 
\frac{5x^3y^2}{x^4 + y^2}, & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0)
\end{cases} \)

47. **Interdisciplinary Problem** Consider measurements and formulas you are using, or have used, in other science or engineering courses. Show how to apply differentials to these measurements and formulas to estimate possible propagated errors.
Section 13.5

### Chain Rules for Functions of Several Variables

- Use the Chain Rules for functions of several variables.
- Find partial derivatives implicitly.

#### Chain Rules for Functions of Several Variables

Your work with differentials in the preceding section provides the basis for the extension of the Chain Rule to functions of two variables. There are two cases—the first case involves a function of \( x \) and \( y \), where \( x \) and \( y \) are functions of a single independent variable \( t \). (A proof of this theorem is given in Appendix A.)

#### THEOREM 13.6 Chain Rule: One Independent Variable

Let \( w = f(x, y) \), where \( f \) is a differentiable function of \( x \) and \( y \). If \( x = g(t) \) and \( y = h(t) \), where \( g \) and \( h \) are differentiable functions of \( t \), then \( w \) is a differentiable function of \( t \), and

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\]

See Figure 13.39.

#### EXAMPLE 1 Using the Chain Rule with One Independent Variable

Let \( w = x^2y - y^2 \), where \( x = \sin t \) and \( y = e^t \). Find \( \frac{dw}{dt} \) when \( t = 0 \).

**Solution** By the Chain Rule for one independent variable, you have

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\]

\[
= 2xy(\cos t) + (x^2 - 2y)e^t
\]

\[
= 2\sin t(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t
\]

\[
= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.
\]

When \( t = 0 \), it follows that

\[
\frac{dw}{dt} = -2.
\]

**Try It**

The Chain Rules presented in this section provide alternative techniques for solving many problems in single-variable calculus. For instance, in Example 1, you could have used single-variable techniques to find \( \frac{dw}{dt} \) by first writing \( w \) as a function of \( t \),

\[
w = x^2y - y^2
\]

\[
= (\sin t)^2(e^t) - (e^t)^2
\]

\[
= e^t \sin^2 t - e^{2t}
\]

and then differentiating as usual.

\[
\frac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}
\]
The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each $x_i$ is a differentiable function of a single variable $t$, then for
\[ w = f(x_1, x_2, \ldots, x_n) \]
you have
\[ \frac{dw}{dt} = \frac{\partial w}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial w}{\partial x_2} \frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n} \frac{dx_n}{dt}. \]

**EXAMPLE 2**  
An Application of a Chain Rule to Related Rates

Two objects are traveling in elliptical paths given by the following parametric equations.

- $x_1 = 4 \cos t$ and $y_1 = 2 \sin t$  
  **First object**
- $x_2 = 2 \sin 2t$ and $y_2 = 3 \cos 2t$  
  **Second object**

At what rate is the distance between the two objects changing when $t = \pi$?

**Solution**  
From Figure 13.40, you can see that the distance $s$ between the two objects is given by
\[ s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
and that when $t = \pi$, you have $x_1 = -4$, $y_1 = 0$, $x_2 = 0$, $y_2 = 3$, and
\[ s = \sqrt{(0 - 4)^2 + (3 - 0)^2} = 5. \]

When $t = \pi$, the partial derivatives of $s$ are as follows.

- $\frac{\partial s}{\partial x_1} = \frac{-(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{-1}{5}(0 + 4) = \frac{-4}{5}$
- $\frac{\partial s}{\partial y_1} = \frac{-(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{-1}{5}(3 - 0) = \frac{-3}{5}$
- $\frac{\partial s}{\partial x_2} = \frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{1}{5}(0 + 4) = \frac{4}{5}$
- $\frac{\partial s}{\partial y_2} = \frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{1}{5}(3 - 0) = \frac{3}{5}$

When $t = \pi$, the derivatives of $x_1$, $y_1$, $x_2$, and $y_2$ are
\[ \frac{dx_1}{dt} = -4 \sin t = 0 \quad \frac{dy_1}{dt} = 2 \cos t = -2 \]
\[ \frac{dx_2}{dt} = 4 \cos 2t = 4 \quad \frac{dy_2}{dt} = -6 \sin 2t = 0. \]

So, using the appropriate Chain Rule, you know that the distance is changing at a rate of
\[ \frac{ds}{dt} = \frac{\partial s}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial s}{\partial y_1} \frac{dy_1}{dt} + \frac{\partial s}{\partial x_2} \frac{dx_2}{dt} + \frac{\partial s}{\partial y_2} \frac{dy_2}{dt} = \left( \frac{-4}{5} \right)(0) + \left( \frac{-3}{5} \right)(-2) + \left( \frac{4}{5} \right)(4) + \left( \frac{3}{5} \right)(0) = \frac{22}{5}. \]
In Example 2, note that \( s \) is the function of four intermediate variables, \( x_1, y_1, x_2, \) and \( y_2 \), each of which is a function of a single variable \( t \). Another type of composite function is one in which the intermediate variables are themselves functions of more than one variable. For instance, if \( w = f(x, y) \), where \( x = g(s, t) \) and \( y = h(s, t) \), it follows that \( w \) is a function of \( s \) and \( t \), and you can consider the partial derivatives of \( w \) with respect to \( s \) and \( t \). One way to find these partial derivatives is to write \( w \) as a function of \( s \) and \( t \) explicitly by substituting the equations \( x = g(s, t) \) and \( y = h(s, t) \) into the equation \( w = f(x, y) \). Then you can find the partial derivatives in the usual way, as demonstrated in the next example.

**Example 3** Finding Partial Derivatives by Substitution

Find \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) for \( w = 2xy \), where \( x = s^2 + t^2 \) and \( y = s/t \).

**Solution** Begin by substituting \( x = s^2 + t^2 \) and \( y = s/t \) into the equation \( w = 2xy \) to obtain

\[
w = 2xy = 2(s^2 + t^2)\left(\frac{s}{t}\right) = 2\left(\frac{s^3}{t} + st\right).
\]

Then, to find \( \frac{\partial w}{\partial s} \), hold \( t \) constant and differentiate with respect to \( s \).

\[
\frac{\partial w}{\partial s} = 2\left(\frac{3s^2}{t} + t\right) = \frac{6s^2 + 2t^2}{t}
\]

Similarly, to find \( \frac{\partial w}{\partial t} \), hold \( s \) constant and differentiate with respect to \( t \) to obtain

\[
\frac{\partial w}{\partial t} = 2\left(\frac{s^3}{t^2} + s\right) = \frac{2s^3 - 2st^2}{t^2}.
\]

**Try It**

Theorem 13.7 gives an alternative method for finding the partial derivatives in Example 3, without explicitly writing \( w \) as a function of \( s \) and \( t \).

**Theorem 13.7 Chain Rule: Two Independent Variables**

Let \( w = f(x, y) \), where \( f \) is a differentiable function of \( x \) and \( y \). If \( x = g(s, t) \) and \( y = h(s, t) \) such that the first partials \( \frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial s}, \) and \( \frac{\partial y}{\partial t} \) all exist, then \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) exist and are given by

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}.
\]

**Proof** To obtain \( \frac{\partial w}{\partial s} \), hold \( t \) constant and apply Theorem 13.6 to obtain the desired result. Similarly, for \( \frac{\partial w}{\partial t} \) hold \( s \) constant and apply Theorem 13.6.

**NOTE** The Chain Rule in this theorem is shown schematically in Figure 13.41.
CHAPTER 13 Functions of Several Variables

You obtain the following.

Similarly, holding constant gives

Theorem 13.7 can also be extended to any number of variables. For example, if \( w \) is a differentiable function of the \( n \) variables \( x_1, x_2, \ldots, x_n \), where each \( x_i \) is a differentiable function of the \( m \) variables \( t_1, t_2, \ldots, t_m \), then for

you obtain the following.

Solution Note that these same partials were found in Example 3. This time, using

Try It
EXAMPLE 5  The Chain Rule for a Function of Three Variables

Find \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) when \( s = 1 \) and \( t = 2\pi \) for the function given by

\[
w = xy + yz + xz
\]

where \( x = s \cos t, y = s \sin t, \) and \( z = t. \)

Solution  By extending the result of Theorem 13.7, you have

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}
\]

\[
= (y + z)(\cos t) + (x + z)(\sin t) + (y + x)(0)
\]

\[
= (y + z)(\cos t) + (x + z)(\sin t).
\]

When \( s = 1 \) and \( t = 2\pi, \) you have \( x = 1, \ y = 0, \) and \( z = 2\pi. \) So, \( \frac{\partial w}{\partial s} = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi. \) Furthermore,

\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}
\]

\[
= (y + z)(-s \sin t) + (x + z)(s \cos t) + (y + x)(1)
\]

and for \( s = 1 \) and \( t = 2\pi, \) it follows that

\[
\frac{\partial w}{\partial t} = (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1)
\]

\[
= 2 + 2\pi.
\]

Try It  Exploration A

Implicit Partial Differentiation

This section concludes with an application of the Chain Rule to determine the derivative of a function defined implicitly. Suppose that \( x \) and \( y \) are related by the equation \( F(x,y) = 0, \) where it is assumed that \( y = f(x) \) is a differentiable function of \( x. \) To find \( dy/dx, \) you could use the techniques discussed in Section 2.5. However, you will see that the Chain Rule provides a convenient alternative. If you consider the function given by

\[
w = F(x, y) = F(x, f(x))
\]

you can apply Theorem 13.6 to obtain

\[
\frac{dw}{dx} = F_x(x, y) \frac{dx}{dx} + F_y(x, y) \frac{dy}{dx}.
\]

Because \( w = F(x, y) = 0 \) for all \( x \) in the domain of \( f, \) you know that \( dw/dx = 0 \) and you have

\[
F_x(x, y) \frac{dx}{dx} + F_y(x, y) \frac{dy}{dx} = 0.
\]

Now, if \( F_x(x, y) \neq 0, \) you can use the fact that \( dx/dx = 1 \) to conclude that

\[
\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.
\]

A similar procedure can be used to find the partial derivatives of functions of several variables that are defined implicitly.
CHAPTER 13 Functions of Several Variables

**THEOREM 13.8 Chain Rule: Implicit Differentiation**

If the equation \( F(x, y) = 0 \) defines \( y \) implicitly as a differentiable function of \( x \), then

\[
\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.
\]

If the equation \( F(x, y, z) = 0 \) defines \( z \) implicitly as a differentiable function of \( x \) and \( y \), then

\[
\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.
\]

This theorem can be extended to differentiable functions defined implicitly with any number of variables.

**EXAMPLE 6 Finding a Derivative Implicitly**

Find \( dy/dx \), given \( y^3 + y^2 - 5y - x^2 + 4 = 0 \).

**Solution** Begin by defining a function \( F \) as

\[ F(x, y) = y^3 + y^2 - 5y - x^2 + 4. \]

Then, using Theorem 13.8, you have

\[ F_x(x, y) = -2x \quad \text{and} \quad F_y(x, y) = 3y^2 + 2y - 5 \]

and it follows that

\[
\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{-2x}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.
\]

**EXAMPLE 7 Finding Partial Derivatives Implicitly**

Find \( \partial z/\partial x \) and \( \partial z/\partial y \), given \( 3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0 \).

**Solution** To apply Theorem 13.8, let

\[ F(x, y, z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5. \]

Then

\[ F_x(x, y, z) = 6xz - 2xy^2 \]
\[ F_y(x, y, z) = -2x^2y + 3z \]
\[ F_z(x, y, z) = 3x^2 + 6z^2 + 3y \]

and you obtain

\[
\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 + 3y} \]
\[
\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{2x^2y - 3z}{3x^2 + 6z^2 + 3y}.
\]
Exercises for Section 13.5

The symbol $\square$ indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on $\square$ to view the complete solution of the exercise.

Click on $\square$ to print an enlarged copy of the graph.

In Exercises 1–4, find $dw/dt$ using the appropriate Chain Rule.

1. $w = x^2 + y^2$
   \[ x = e^t, \ y = e^{-t} \]
2. $w = \sqrt{x^2 + y^2}$
   \[ x = \cos t, \ y = e^t \]
3. $w = x \sec y$
   \[ x = e^t, \ y = \pi - t \]
4. $w = \ln \frac{y}{x}$
   \[ x = \cos t, \ y = \sin t \]

In Exercises 5–10, find $dw/dt$ (a) using the appropriate Chain Rule and (b) by converting $w$ to a function of $t$ before differentiating.

5. $w = xy$, \quad $x = 2 \sin t$, \quad $y = \cos t$
6. $w = \cos(x - y)$, \quad $x = r^2$, \quad $y = 1$
7. $w = x^2 + y^2 + z^2$, \quad $x = e^t \cos t$, \quad $y = e^t \sin t$, \quad $z = e^t$
8. $w = xy \cos z$, \quad $x = t$, \quad $y = t^2$, \quad $z = \arccos t$
9. $w = xy + xz + yz$, \quad $x = t - 1$, \quad $y = t^2 - 1$, \quad $z = t$
10. $w = xyz$, \quad $x = t^2$, \quad $y = 2t$, \quad $z = e^{-t}$

**Projectile Motion** In Exercises 11 and 12, the parametric equations for the paths of two projectiles are given. At what rate is the distance between the two objects changing at the given value of $t$?

11. $x_1 = 10 \cos 2t$, $y_1 = 6 \sin 2t$ \quad First object
   \[ x_2 = 7 \cos t, \ y_2 = 4 \sin t \quad Second object \]
   \[ t = \pi/2 \]
12. $x_1 = 48 \sqrt{3}t$, $y_1 = 48 \sqrt{2}t - 16t^2$ \quad First object
   \[ x_2 = 48 \sqrt{3}t$, $y_2 = 48t - 16t^2 \quad Second object \]
   \[ t = 1 \]

In Exercises 13 and 14, find $d^2w/dt^2$ using the appropriate Chain Rule. Evaluate $d^2w/dt^2$ at the given value of $t$.

13. $w = \arctan(2xy)$, \quad $x = \cos t$, \quad $y = \sin t$, \quad $t = 0$
14. $w = \frac{x^2}{y}$, \quad $x = t^2$, \quad $y = t + 1$, \quad $t = 1$

In Exercises 15–18, find $\partial w/\partial s$ and $\partial w/\partial t$ using the appropriate Chain Rule, and evaluate each partial derivative at the given values of $s$ and $t$.

<table>
<thead>
<tr>
<th>Function</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w = x^2 + y^2$</td>
<td>$s = 2$, $t = -1$</td>
</tr>
<tr>
<td>$w = x^3 + 3xy^2$</td>
<td>$s = 0$, $t = 1$</td>
</tr>
<tr>
<td>$w = x^3 + y$</td>
<td>$s = 3$, $t = \frac{\pi}{4}$</td>
</tr>
<tr>
<td>$w = \sin(2x + 3y)$</td>
<td>$s = 0$, $t = \frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

In Exercises 19–22, find $\partial w/\partial r$ and $\partial w/\partial \theta$ (a) using the appropriate Chain Rule and (b) by converting $w$ to a function of $r$ and $\theta$ before differentiating.

19. $w = x^2 - 2xy + y^2$, \quad $x = r + \theta$, \quad $y = r - \theta$
20. $w = \sqrt{25 - 5x^2 - 5y^2}$, \quad $x = r \cos \theta$, \quad $y = r \sin \theta$
21. $w = \arctan \frac{y}{x}$, \quad $x = r \cos \theta$, \quad $y = r \sin \theta$
22. $w = \frac{yz}{x}$, \quad $x = \theta^2$, \quad $y = r + \theta$, \quad $z = r - \theta$

In Exercises 23–26, find $\partial w/\partial s$ and $\partial w/\partial t$ by using the appropriate Chain Rule.

23. $w = xyz$, \quad $x = s + t$, \quad $y = s - t$, \quad $z = st^2$
24. $w = x \cos yz$, \quad $x = s^2$, \quad $y = t^2$, \quad $z = s - 2t$
25. $w = ze^{\sqrt{s}}$, \quad $x = s - t$, \quad $y = s + t$, \quad $z = st$
26. $w = x^2 + y^2 + z^2$, \quad $x = t \sin s$, \quad $y = t \cos s$, \quad $z = st^2$

In Exercises 27–30, differentiate implicitly to find $dy/dx$.

27. $x^2 - 3xy + y^2 - 2x + y - 5 = 0$
28. $\cos x + \tan xy + 5 = 0$
29. $\ln \sqrt{x^2 + y^2} + xy = 4$
30. $\frac{x}{x^2 + y^2} - y^2 = 6$

In Exercises 31–38, differentiate implicitly to find the first partial derivatives of $z$.

31. $x^2 + y^2 + z^2 = 25$
32. $xz + yz + xy = 0$
33. $\tan(x + y) + \tan(y + z) = 1$
34. $z = e^x \sin(y + z)$
35. $x^2 + 2yz + z^2 = 1$
36. $x + \sin(y + z) = 0$
37. $e^{xt} + xy = 0$
38. $x \ln y + y^2z + z^2 = 8$

In Exercises 39–42, differentiate implicitly to find the first partial derivatives of $w$.

39. $xyz + xzw - yzw + w^2 = 5$
40. $x^2 + y^2 + z^2 - 5yw + 10w^2 = 2$
41. $\cos xy + \sin yz + wz = 20$
42. $w - \sqrt{x - y - \sqrt{y + z}} = 0$

**Homogeneous Functions** In Exercises 43–46, the function $f$ is homogeneous of degree $n$ if $f(tx, ty) = t^n f(x, y)$. Determine the degree of the homogeneous function, and show that $xf(x, y) + yf(x, y) = nf(x, y)$.

43. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$
44. $f(x, y) = x^3 - 3xy^2 + y^3$
45. $f(x, y) = e^{x/y}$
46. $f(x, y) = \frac{x^2}{\sqrt{x^2 + y^2}}$
51. **Volume and Surface Area**  The radius of a right circular cylinder is increasing at a rate of 6 inches per minute, and the height is decreasing at a rate of 4 inches per minute. What are the rates of change of the volume and surface area when the radius is 12 inches and the height is 36 inches?

52. **Volume and Surface Area**  Repeat Exercise 51 for a right circular cone.

53. **Area**  Let \( \theta \) be the angle between equal sides of an isosceles triangle and let \( x \) be the length of these sides. \( x \) is increasing at \( \frac{1}{2} \) meter per hour and \( \theta \) is increasing at \( \pi/90 \) radian per hour. Find the rate of increase of the area when \( x = 6 \) and \( \theta = \pi/4 \).

54. **Volume and Surface Area**  The two radii of the frustum of a right circular cone are increasing at a rate of 4 centimeters per minute, and the height is increasing at a rate of 12 centimeters per minute (see figure). Find the rates at which the volume and surface area are changing when the two radii are 15 centimeters and 25 centimeters, and the height is 10 centimeters.

55. **Moment of Inertia**  An annular cylinder has an inside radius of \( r_1 \) and an outside radius of \( r_2 \) (see figure). Its moment of inertia is

\[
I = \frac{1}{2}m(r_1^2 + r_2^2)
\]

where \( m \) is the mass. The two radii are increasing at a rate of 2 centimeters per second. Find the rate at which \( I \) is changing at the instant the radii are 6 centimeters and 8 centimeters. (Assume mass is constant.)

56. **Ideal Gas Law**  The Ideal Gas Law is \( pV = mRT \), where \( R \) is a constant, \( m \) is a constant mass, and \( p \) and \( V \) are functions of time. Find \( dT/dt \), the rate at which the temperature changes with respect to time.

57. **Maximum Angle**  A two-foot-tall painting hangs on a wall such that the bottom is 6 feet from the floor. A child whose eye are 4 feet above the floor stands \( x \) feet from the wall (see figure).

(a) Show that \( x^2 \tan \theta - 2x + 8 \tan \theta = 0 \).

(b) Use implicit differentiation to find \( d\theta/dx \).

(c) Find \( x \) such that \( \theta \) is maximum.

58. Show that if \( f(x,y) \) is homogeneous of degree \( n \), then

\[
x^n f(x, y) + y^n f(x, y) = n f(x, y).
\]

[Hint: Let \( g(t) = f(tx, ty) = t^n f(x, y) \). Find \( g'(t) \) and then let \( t = 1 \).]

59. Show that

\[
\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0
\]

for \( w = f(x, y), x = u - v, \) and \( y = v - u \).

60. Demonstrate the result of Exercise 59 for \( w = (x - y) \sin(y - x) \).

61. Consider the function \( w = f(x, y) \), where \( x = r \cos \theta \) and \( y = r \sin \theta \). Prove each of the following.

(a) \( \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cos \theta - \frac{\partial w}{\partial \theta} \frac{\sin \theta}{r} \)

(b) \( \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = \left( \frac{\partial w}{\partial r} \right)^2 + \left( \frac{1}{r} \right) \left( \frac{\partial w}{\partial \theta} \right)^2 \)

62. Demonstrate the result of Exercise 61(b) for \( w = \arctan(y/x) \).

63. **Cauchy-Riemann Equations**  Given the functions \( u(x, y) \) and \( v(x, y) \), verify that the Cauchy-Riemann differential equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

can be written in polar coordinate form as

\[
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}
\]

64. Demonstrate the result of Exercise 63 for the functions

\[
u = \ln \sqrt{x^2 + y^2} \quad \text{and} \quad v = \arctan \frac{y}{x}
\]
### Section 13.6

**Directional Derivatives and Gradients**

- Find and use directional derivatives of a function of two variables.
- Find the gradient of a function of two variables.
- Use the gradient of a function of two variables in applications.
- Find directional derivatives and gradients of functions of three variables.

#### Directional Derivative

You are standing on the hillside pictured in Figure 13.42 and want to determine the hill’s incline toward the z-axis. If the hill were represented by \( z = f(x, y) \), you would already know how to determine the slopes in two different directions—the slope in the y-direction would be given by the partial derivative \( f_y(x, y) \), and the slope in the x-direction would be given by the partial derivative \( f_x(x, y) \). In this section, you will see that these two partial derivatives can be used to find the slope in any direction.

To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative**. Begin by letting \( z = f(x, y) \) be a surface and \( P(x_0, y_0) \) a point in the domain of \( f \), as shown in Figure 13.43. The “direction” of the directional derivative is given by a unit vector

\[
\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}
\]

where \( \theta \) is the angle the vector makes with the positive x-axis. To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point \( P \) and parallel to \( \mathbf{u} \), as shown in Figure 13.44. This vertical plane intersects the surface to form a curve \( C \). The slope of the surface at \( (x_0, y_0, f(x_0, y_0)) \) in the direction of \( \mathbf{u} \) is defined as the slope of the curve \( C \) at that point.

Informally, you can write the slope of the curve \( C \) as a limit that looks much like those used in single-variable calculus. The vertical plane used to form \( C \) intersects the \( xy \)-plane in a line \( L \), represented by the parametric equations

\[
x = x_0 + t \cos \theta
\]

and

\[
y = y_0 + t \sin \theta
\]

so that for any value of \( t \), the point \( Q(x, y) \) lies on the line \( L \). For each of the points \( P \) and \( Q \), there is a corresponding point on the surface.

\[
(x_0, y_0, f(x_0, y_0)) \quad \text{Point above } P
\]

\[
(x, y, f(x, y)) \quad \text{Point above } Q
\]

Moreover, because the distance between \( P \) and \( Q \) is

\[
\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(t \cos \theta)^2 + (t \sin \theta)^2} = |t|
\]

you can write the slope of the secant line through \( (x_0, y_0, f(x_0, y_0)) \) and \( (x, y, f(x, y)) \) as

\[
f(x, y) - f(x_0, y_0) \quad \frac{t}{t} = \frac{f(x_0 + t \cos \theta, y_0 + t \sin \theta) - f(x_0, y_0)}{t}.
\]

Finally, by letting \( t \) approach 0, you arrive at the following definition.
**Definition of Directional Derivative**

Let $f$ be a function of two variables $x$ and $y$ and let $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ be a unit vector. Then the **directional derivative of $f$ in the direction of $\mathbf{u}$**, denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t \to 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

provided this limit exists.

Calculating directional derivatives by this definition is similar to finding the derivative of a function of one variable by the limit process (given in Section 2.1). A simpler “working” formula for finding directional derivatives involves the partial derivatives $f_x$ and $f_y$.

**THEOREM 13.9 Directional Derivative**

If $f$ is a differentiable function of $x$ and $y$, then the directional derivative of $f$ in the direction of the unit vector $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

**Proof**  For a fixed point $(x_0, y_0)$, let $x = x_0 + t \cos \theta$ and let $y = y_0 + t \sin \theta$. Then, let $g(t) = f(x, y)$. Because $f$ is differentiable, you can apply the Chain Rule given in Theorem 13.7 to obtain

$$g'(t) = f_x(x, y)x'(t) + f_y(x, y)y'(t) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$  

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0) \cos \theta + f_y(x_0, y_0) \sin \theta.$$  

By the definition of $g'(t)$, it is also true that

$$g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{f(x_0 + t \cos \theta, y_0 + t \sin \theta) - f(x_0, y_0)}{t}.$$  

Consequently, $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0) \cos \theta + f_y(x_0, y_0) \sin \theta$.

There are infinitely many directional derivatives to a surface at a given point—one for each direction specified by $\mathbf{u}$, as shown in Figure 13.45. Two of these are the partial derivatives $f_x$ and $f_y$.

1. **Direction of positive $x$-axis ($\theta = 0$):** $\mathbf{u} = \cos 0 \mathbf{i} + \sin 0 \mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}}f(x, y) = f_x(x, y) \cos 0 + f_y(x, y) \sin 0 = f_x(x, y)$$

2. **Direction of positive $y$-axis ($\theta = \pi/2$):** $\mathbf{u} = \cos \pi/2 \mathbf{i} + \sin \pi/2 \mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}}f(x, y) = f_x(x, y) \cos \pi/2 + f_y(x, y) \sin \pi/2 = f_y(x, y)$$
**Example 1** Finding a Directional Derivative

Find the directional derivative of

\[ f(x, y) = 4 - x^2 - \frac{1}{2}y^2 \]

at \((1, 2)\) in the direction of

\[ \mathbf{u} = \left( \cos \frac{\pi}{3} \right) \mathbf{i} + \left( \sin \frac{\pi}{3} \right) \mathbf{j}. \]

**Solution**  Because \(f_x\) and \(f_y\) are continuous, \(f\) is differentiable, and you can apply Theorem 13.9.

\[
D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta
\]

Evaluating at \(\theta = \pi/3, x = 1,\) and \(y = 2\) produces

\[
D_{\mathbf{u}} f(1, 2) = (-2)(1) + (1)\left(\frac{\sqrt{3}}{2}\right)
= -1 - \frac{\sqrt{3}}{2}
\approx -1.866.
\]

You have been specifying direction by a unit vector \(\mathbf{u}\). If the direction is given by a vector whose length is not 1, you must normalize the vector before applying the formula in Theorem 13.9.

**Example 2** Finding a Directional Derivative

Find the directional derivative of

\[ f(x, y) = x^2 \sin 2y \]

at \((1, \pi/2)\) in the direction of

\[ \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}. \]

**Solution**  Because \(f_x\) and \(f_y\) are continuous, \(f\) is differentiable, and you can apply Theorem 13.9. Begin by finding a unit vector in the direction of \(\mathbf{v}\).

\[
\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{3}{5} \mathbf{i} - \frac{4}{5} \mathbf{j} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}
\]

Using this unit vector, you have

\[
D_{\mathbf{u}} f(1, \pi/2) = (2x \sin 2y)(\cos \theta) + (2x^2 \cos 2y)(\sin \theta)
\]

\[
D_{\mathbf{u}} f(1, \pi/2) = (2 \cdot 1 \cdot \sin \pi\left(\frac{3}{5}\right)) + (2 \cdot \cos \pi\left(\frac{-4}{5}\right))
= 0 \cdot \left(\frac{3}{5}\right) + (-2) \cdot \left(\frac{-4}{5}\right)
= \frac{8}{5}.
\]

You have been specifying direction by a unit vector \(\mathbf{u}\). If the direction is given by a vector whose length is not 1, you must normalize the vector before applying the formula in Theorem 13.9.
The Gradient of a Function of Two Variables

The gradient of a function of two variables is a vector-valued function of two variables. This function has many important uses, some of which are described later in this section.

**Definition of Gradient of a Function of Two Variables**

Let \( z = f(x, y) \) be a function of \( x \) and \( y \) such that \( f_x \) and \( f_y \) exist. Then the gradient of \( f \), denoted by \( \nabla f(x, y) \), is the vector

\[
\nabla f(x, y) = f_x(x, y)i + f_y(x, y)j.
\]

**NOTE** No value is assigned to the symbol \( \nabla \) by itself. It is an operator in the same sense that \( d/\,dx \) is an operator. When \( \nabla \) operates on \( f(x, y) \), it produces the vector \( \nabla f(x, y) \).

**EXAMPLE 3** Finding the Gradient of a Function

Find the gradient of \( f(x, y) = y \ln x + xy^2 \) at the point \((1, 2)\).

**Solution** Using

\[
f_x(x, y) = \frac{y}{x} + y^2 \quad \text{and} \quad f_y(x, y) = \ln x + 2xy
\]

you have

\[
\nabla f(x, y) = \left(\frac{y}{x} + y^2\right)i + (\ln x + 2xy)j.
\]

At the point \((1, 2)\), the gradient is

\[
\nabla f(1, 2) = \left(\frac{2}{1} + 2^2\right)i + (\ln 1 + 2(1)(2))j
\]

\[
= 6i + 4j.
\]

**Try It**

Because the gradient of \( f \) is a vector, you can write the directional derivative of \( f \) in the direction of \( u \) as

\[
D_u f(x, y) = [f_x(x, y)i + f_y(x, y)j] \cdot [\cos \theta i + \sin \theta j].
\]

In other words, the directional derivative is the dot product of the gradient and the direction vector. This useful result is summarized in the following theorem.

**THEOREM 13.10** Alternative Form of the Directional Derivative

If \( f \) is a differentiable function of \( x \) and \( y \), then the directional derivative of \( f \) in the direction of the unit vector \( u \) is

\[
D_u f(x, y) = \nabla f(x, y) \cdot u.
\]
EXAMPLE 4 Using \( \nabla f(x, y) \) to Find a Directional Derivative

Find the directional derivative of 
\[
f(x, y) = 3x^2 - 2y^2
\]
at \((-\frac{3}{4}, 0)\) in the direction from \(P(-\frac{3}{4}, 0)\) to \(Q(0, 1)\).

Solution  Because the partials of \( f \) are continuous, \( f \) is differentiable and you can apply Theorem 13.10. A vector in the specified direction is 
\[
\overrightarrow{PQ} = \mathbf{v} = \left( 0 + \frac{3}{4} \right) \mathbf{i} + (1 - 0) \mathbf{j} = \frac{3}{4} \mathbf{i} + \mathbf{j}
\]
and a unit vector in this direction is 
\[
\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}.
\]
Because \( \nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j} = 6xi - 4yj \), the gradient at \((-\frac{3}{4}, 0)\) is 
\[
\nabla f \left(-\frac{3}{4}, 0\right) = -\frac{9}{2} \mathbf{i} + 0 \mathbf{j}.
\]
Consequently, at \((-\frac{3}{4}, 0)\) the directional derivative is 
\[
D_u f \left(-\frac{3}{4}, 0\right) = \nabla f \left(-\frac{3}{4}, 0\right) \cdot \mathbf{u} = \left( -\frac{9}{2} \mathbf{i} + 0 \mathbf{j} \right) \cdot \left( \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j} \right) = -\frac{27}{10}.
\]
See Figure 13.49.

Applications of the Gradient

You have already seen that there are many directional derivatives at the point \((x, y)\) on a surface. In many applications, you may want to know in which direction to move so that \( f(x, y) \) increases most rapidly. This direction is called the direction of steepest ascent, and it is given by the gradient, as stated in the following theorem.

THEOREM 13.11 Properties of the Gradient

Let \( f \) be differentiable at the point \((x, y)\).

1. If \( \nabla f(x, y) = 0 \), then \( D_u f(x, y) = 0 \) for all \( \mathbf{u} \).
2. The direction of maximum increase of \( f \) is given by \( \nabla f(x, y) \). The maximum value of \( D_u f(x, y) \) is \( \|\nabla f(x, y)\| \).
3. The direction of minimum increase of \( f \) is given by \(-\nabla f(x, y)\). The minimum value of \( D_u f(x, y) \) is \(-\|\nabla f(x, y)\| \).

NOTE  Part 2 of Theorem 13.11 says that at the point \((x, y)\), \( f \) increases most rapidly in the direction of the gradient, \( \nabla f(x, y) \).
The direction of most rapid increase in temperature is given by \( \langle -16, 6 \rangle \). Figure 13.51

**Proof** If \( \nabla f(x, y) = \mathbf{0} \), then for any direction (any \( \mathbf{u} \)), you have

\[
D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = (0i + 0j) \cdot (\cos \theta i + \sin \theta j) = 0.
\]

If \( \nabla f(x, y) \neq \mathbf{0} \), then let \( \phi \) be the angle between \( \nabla f(x, y) \) and a unit vector \( \mathbf{u} \). Using the dot product, you can apply Theorem 11.5 to conclude that

\[
D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = ||\nabla f(x, y)|| ||\mathbf{u}|| \cos \phi = ||\nabla f(x, y)|| \cos \phi
\]

and it follows that the maximum value of \( D_u f(x, y) \) will occur when \( \cos \phi = 1 \). So, \( \phi = 0 \), and the maximum value for the directional derivative occurs when \( \mathbf{u} \) has the same direction as \( \nabla f(x, y) \). Moreover, this largest value for \( D_u f(x, y) \) is precisely

\[
||\nabla f(x, y)|| \cos \phi = ||\nabla f(x, y)||,
\]

Similarly, the minimum value of \( D_u f(x, y) \) can be obtained by letting \( \phi = \pi \) so that \( \mathbf{u} \) points in the direction opposite that of \( \nabla f(x, y) \), as shown in Figure 13.50.

To visualize one of the properties of the gradient, imagine a skier coming down a mountainside. If \( f(x, y) \) denotes the altitude of the skier, then \( -\nabla f(x, y) \) indicates the compass direction the skier should take to ski the path of steepest descent. (Remember that the gradient indicates direction in the \( xy \)-plane and does not itself point up or down the mountainside.)

As another illustration of the gradient, consider the temperature \( T(x, y) \) at any point \( (x, y) \) on a flat metal plate. In this case, \( \nabla T(x, y) \) gives the direction of greatest temperature increase at the point \( (x, y) \), as illustrated in the next example.

**EXAMPLE 5** Finding the Direction of Maximum Increase

The temperature in degrees Celsius on the surface of a metal plate is

\[
T(x, y) = 20 - 4x^2 - y^2
\]

where \( x \) and \( y \) are measured in centimeters. In what direction from \( (2, -3) \) does the temperature increase most rapidly? What is this rate of increase?

**Solution** The gradient is

\[
\nabla T(x, y) = T_x(x, y)i + T_y(x, y)j = -8xi - 2yj.
\]

It follows that the direction of maximum increase is given by

\[
\nabla T(2, -3) = -16i + 6j
\]

as shown in Figure 13.51, and the rate of increase is

\[
||\nabla T(2, -3)|| = \sqrt{256 + 36} = \sqrt{292} \approx 17.09^\circ \text{ per centimeter.}
\]
The solution presented in Example 5 can be misleading. Although the gradient points in the direction of maximum temperature increase, it does not necessarily point toward the hottest spot on the plate. In other words, the gradient provides a local solution to finding an increase relative to the temperature at the point \((2, -3)\). Once you leave that position, the direction of maximum increase may change.

**EXAMPLE 6  Finding the Path of a Heat-Seeking Particle**

A heat-seeking particle is located at the point \((2, -3)\) on a metal plate whose temperature at \((x, y)\) is

\[
T(x, y) = 20 - 4x^2 - y^2.
\]

Find the path of the particle as it continuously moves in the direction of maximum temperature increase.

**Solution**  Let the path be represented by the position function

\[
r(t) = x(t)i + y(t)j.
\]

A tangent vector at each point \((x(t), y(t))\) is given by

\[
r'(t) = \frac{dx}{dt}i + \frac{dy}{dt}j.
\]

Because the particle seeks maximum temperature increase, the directions of \(r'(t)\) and \(\nabla T(x, y) = -8x - 2yj\) are the same at each point on the path. So,

\[
-8x = k\frac{dx}{dt} \quad \text{and} \quad -2y = k\frac{dy}{dt}
\]

where \(k\) depends on \(t\). By solving each equation for \(dt/k\) and equating the results, you obtain

\[
\frac{dx}{-8x} = \frac{dy}{-2y}.
\]

The solution of this differential equation is \(x = Cy^4\). Because the particle starts at the point \((2, -3)\), you can determine that \(C = 2/81\). So, the path of the heat-seeking particle is

\[
x = \frac{2}{81}y^4.
\]

The path is shown in Figure 13.52.

**Try It**  In Figure 13.52, the path of the particle (determined by the gradient at each point) appears to be orthogonal to each of the level curves. This becomes clear when you consider that the temperature \(T(x, y)\) is constant along a given level curve. So, at any point \((x, y)\) on the curve, the rate of change of \(T\) in the direction of a unit tangent vector \(u\) is 0, and you can write

\[
\nabla f(x, y) \cdot u = D_u T(x, y) = 0. \quad u \text{ is a unit tangent vector.}
\]

Because the dot product of \(\nabla f(x, y)\) and \(u\) is 0, you can conclude that they must be orthogonal. This result is stated in the following theorem.
EXAMPLE 7  Finding a Normal Vector to a Level Curve

Sketch the level curve corresponding to \( c = 0 \) for the function given by

\[
f(x, y) = y - \sin x
\]

and find a normal vector at several points on the curve.

Solution  The level curve for \( c = 0 \) is given by

\[
0 = y - \sin x
\]

\[
y = \sin x
\]

as shown in Figure 13.53(a). Because the gradient vector of \( f \) at \((x, y)\) is

\[
\nabla f(x, y) = f_x(x, y)i + f_y(x, y)j
\]

\[
= -\cos x i + j
\]

you can use Theorem 13.12 to conclude that \( \nabla f(x, y) \) is normal to the level curve at the point \((x, y)\). Some gradient vectors are

\[
\nabla f(-\pi, 0) = i + j
\]

\[
\nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}i + j
\]

\[
\nabla f\left(-\frac{\pi}{2}, -1\right) = j
\]

\[
\nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}i + j
\]

\[
\nabla f(0, 0) = -i + j
\]

\[
\nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}i + j
\]

\[
\nabla f\left(\frac{\pi}{2}, 1\right) = j.
\]

These are shown in Figure 13.53(b).

![Figure 13.53](image)

(a) The surface is given by \( f(x, y) = y - \sin x \).

(b) The level curve is given by \( f(x, y) = 0 \).
Functions of Three Variables

The definitions of the directional derivative and the gradient can be extended naturally to functions of three or more variables. As often happens, some of the geometric interpretation is lost in the generalization from functions of two variables to those of three variables. For example, you cannot interpret the directional derivative of a function of three variables to represent slope.

The definitions and properties of the directional derivative and the gradient of a function of three variables are given in the following summary.

### Directional Derivative and Gradient for Three Variables

Let $f$ be a function of $x$, $y$, and $z$, with continuous first partial derivatives. The **directional derivative of $f$** in the direction of a unit vector $\mathbf{u} = ai + bj + ck$ is given by

$$ D_\mathbf{u}f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z). $$

The **gradient of $f$** is defined to be

$$ \nabla f(x, y, z) = f_x(x, y, z)i + f_y(x, y, z)j + f_z(x, y, z)k. $$

Properties of the gradient are as follows.

1. $D_\mathbf{u}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = 0$, then $D_\mathbf{u}f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of **maximum** increase of $f$ is given by $\nabla f(x, y, z)$. The maximum value of $D_\mathbf{u}f(x, y, z)$ is

$$ \|\nabla f(x, y, z)\|. \quad \text{Maximum value of } D_\mathbf{u}f(x, y, z) $$

4. The direction of **minimum** increase of $f$ is given by $-\nabla f(x, y, z)$. The minimum value of $D_\mathbf{u}f(x, y, z)$ is

$$ -\|\nabla f(x, y, z)\|. \quad \text{Minimum value of } D_\mathbf{u}f(x, y, z) $$

**NOTE** You can generalize Theorem 13.12 to functions of three variables. Under suitable hypotheses,

$$ \nabla f(x_0, y_0, z_0) $$

is normal to the level surface through $(x_0, y_0, z_0)$.

**EXAMPLE 8 Finding the Gradient for a Function of Three Variables**

Find $\nabla f(x, y, z)$ for the function given by

$$ f(x, y, z) = x^2 + y^2 - 4z $$

and find the direction of maximum increase of $f$ at the point $(2, -1, 1)$.

**Solution** The gradient vector is given by

$$ \nabla f(x, y, z) = f_x(x, y, z)i + f_y(x, y, z)j + f_z(x, y, z)k $$

$$ = 2xi + 2yj - 4k. $$

So, it follows that the direction of maximum increase at $(2, -1, 1)$ is

$$ \nabla f(2, -1, 1) = 4i - 2j - 4k. $$
Exercises for Section 13.6

The symbol \(\text{\ding{211}}\) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on \(\text{S}\) to view the complete solution of the exercise.

Click on \(\text{M}\) to print an enlarged copy of the graph.

In Exercises 1–12, find the directional derivative of the function at \(P\) in the direction of \(v\).

1. \(f(x, y) = 3x - 4xy + 5y, \quad P(1, 2), \quad v = \frac{1}{2}(1 + \sqrt{3}j)\)
2. \(f(x, y) = x^3 - y^3, \quad P(4, 3), \quad v = \frac{\sqrt{7}}{2}(i + j)\)
3. \(f(x, y) = xy, \quad P(2, 3), \quad v = i + j\)
4. \(f(x, y) = \frac{x}{y}, \quad P(1, 1), \quad v = -j\)
5. \(g(x, y) = \sqrt{x^2 + y^2}, \quad P(3, 4), \quad v = 3i - 4j\)
6. \(g(x, y) = \arccos xy, \quad P(1, 0), \quad v = i + 5j\)
7. \(h(x, y) = e^x \sin y, \quad P\left(1, \frac{\pi}{2}\right), \quad v = -i\)
8. \(h(x, y) = e^{-(x^2+y^2)}, \quad P(0, 0), \quad v = i + j\)
9. \(f(x, y, z) = xy + yz + xz, \quad P(1, 1, 1), \quad v = 2i + j - k\)
10. \(f(x, y, z) = x^2 + y^2 + z^2, \quad P(1, 2, -1), \quad v = i - 2j + 3k\)
11. \(h(x, y, z) = x \arctan yz, \quad P(4, 1, 1), \quad v = (1, 2, -1)\)
12. \(h(x, y, z) = xyz, \quad P(2, 1, 1), \quad v = (2, 1, 2)\)

In Exercises 13–16, find the directional derivative of the function in the direction of \(u = \cos \theta i + \sin \theta j\).

13. \(f(x, y) = x^2 + y^2, \quad \theta = \frac{\pi}{4}\)
14. \(f(x, y) = \frac{y}{x + y}, \quad \theta = -\frac{\pi}{6}\)
15. \(f(x, y) = \sin(2x - y), \quad \theta = -\frac{\pi}{3}\)
16. \(g(x, y) = xe^\theta, \quad \theta = \frac{2\pi}{3}\)

In Exercises 17–20, find the directional derivative of the function at \(P\) in the direction of \(Q\).

17. \(f(x, y) = x^2 + 4y^2, \quad P(3, 1), \quad Q(1, -1)\)
18. \(f(x, y) = \cos(x + y), \quad P(0, \pi), \quad Q\left(\frac{\pi}{2}, 0\right)\)
19. \(h(x, y, z) = \ln(x + y + z), \quad P(1, 0, 0), \quad Q(4, 3, 1)\)
20. \(g(x, y, z) = xye^z, \quad P(2, 4, 0), \quad Q(0, 0, 0)\)

In Exercises 21–26, find the gradient of the function at the given point.

21. \(f(x, y) = 3x - 5y^2 + 10, \quad (2, 1)\)
22. \(g(x, y) = 2xe^{-x^2}, \quad (2, 0)\)
23. \(z = \cos(x^2 + y^2), \quad (3, -4)\)
24. \(z = \ln(x^2 - y), \quad (2, 3)\)
25. \(w = 3x^2y - 5yz + z^2, \quad (1, 1, -2)\)
26. \(w = x \tan(y + z), \quad (4, 3, -1)\)

In Exercises 27–30, use the gradient to find the directional derivative of the function at \(P\) in the direction of \(Q\).

27. \(g(x, y) = x^2 + y^2 + 1, \quad P(1, 2), \quad Q(3, 6)\)
28. \(f(x, y) = x^3 - y^2 + 4, \quad P(3, 1), \quad Q(1, 8)\)
29. \(f(x, y) = e^{-x} \cos y, \quad P(0, 0), \quad Q(2, 1)\)
30. \(f(x, y) = \sin 2x \cos y, \quad P(0, 0), \quad Q\left(\frac{\pi}{2}, \pi\right)\)

In Exercises 31–38, find the gradient of the function and the maximum value of the directional derivative at the given point.

<table>
<thead>
<tr>
<th>Function</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x, y) = x \tan y)</td>
<td>((2, \frac{\pi}{4}))</td>
</tr>
<tr>
<td>(h(x, y) = y \cos(x - y))</td>
<td>((0, \frac{\pi}{3}))</td>
</tr>
<tr>
<td>(g(x, y) = \ln \sqrt{x^2 + y^2})</td>
<td>((1, 2))</td>
</tr>
<tr>
<td>(g(x, y) = ye^{-x})</td>
<td>((0, 5))</td>
</tr>
<tr>
<td>(f(x, y, z) = \sqrt{x^2 + y^2 + z^2})</td>
<td>((1, 4, 2))</td>
</tr>
<tr>
<td>(w = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}})</td>
<td>((0, 0, 0))</td>
</tr>
<tr>
<td>(f(x, y, z) = xe^{yz})</td>
<td>((2, 0, -4))</td>
</tr>
<tr>
<td>(w = xy^2z^2)</td>
<td>((2, 1, 1))</td>
</tr>
</tbody>
</table>

In Exercises 39–46, use the function \(f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}\).

39. Sketch the graph of \(f\) in the first octant and plot the point \((3, 2, 1)\) on the surface.
40. Find \(D_u f(3, 2)\), where \(u = \cos \theta i + \sin \theta j\).
   (a) \(\theta = \frac{\pi}{4}\)
   (b) \(\theta = \frac{2\pi}{3}\)
41. Find \(D_u f(3, 2)\), where \(u = \cos \theta i + \sin \theta j\).
   (a) \(\theta = \frac{4\pi}{3}\)
   (b) \(\theta = -\frac{\pi}{6}\)
42. Find \(D_u f(3, 2)\), where \(u = \frac{v}{|v|}\).
   (a) \(v = i + j\)
   (b) \(v = -3i - 4j\)
43. Find \(D_u f(3, 2)\), where \(u = \frac{v}{|v|}\).
   (a) \(v\) is the vector from \((1, 2)\) to \((-2, 6)\).
   (b) \(v\) is the vector from \((3, 2)\) to \((4, 5)\).
44. Find \(\nabla f(x, y)\).
45. Find the maximum value of the directional derivative at \((3, 2)\).
46. Find a unit vector \(u\) orthogonal to \(\nabla f(3, 2)\) and calculate \(D_u f(3, 2)\). Discuss the geometric meaning of the result.
In Exercises 47–50, use the function $f(x, y) = 9 - x^2 - y^2$.

47. Sketch the graph of $f$ in the first octant and plot the point $(1, 2, 4)$ on the surface.

48. Find $D_u f(1, 2)$, where $u = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$.

(a) $\theta = -\frac{\pi}{4}$  
(b) $\theta = \frac{\pi}{3}$

49. Find $\nabla f(1, 2)$ and $\|\nabla f(1, 2)\|$.

50. Find a unit vector $u$ orthogonal to $\nabla f(1, 2)$ and calculate $D_u f(1, 2)$. Discuss the geometric meaning of the result.

Investigation In Exercises 51 and 52, (a) use the graph to estimate the components of the vector in the direction of the maximum rate of increase in the function at the given point. (b) Find the gradient at the point and compare it with your estimate in part (a). (c) In what direction would the function be decreasing at the greatest rate? Explain.

51. $f(x, y) = \frac{1}{2}(x^2 - 3xy + y^2)$,  
(1, 2)

52. $f(x, y) = \frac{1}{2}y\sqrt{x}$,  
(1, 2)

53. Investigation Consider the function $f(x, y) = x^2 - y^2$ at the point $(4, -3, 7)$.

(a) Use a computer algebra system to graph the surface represented by the function.

(b) Determine the directional derivative $D_u f(4, -3)$ as a function of $\theta$, where $u = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$. Use a computer algebra system to graph the function on the interval $[0, 2\pi]$.

(c) Approximate the zeros of the function in part (b) and interpret each in the context of the problem.

(d) Approximate the critical numbers of the function in part (b) and interpret each in the context of the problem.

(e) Find $\|\nabla f(4, -3)\|$ and explain its relationship to your answers in part (d).

(f) Use a computer algebra system to graph the level curve of the function $f$ at the level $c = 7$. On this curve, graph the vector in the direction of $\nabla f(4, -3)$, and state its relationship to the level curve.

54. Investigation The figure below shows the level curve of the function $f(x, y) = \frac{8y}{1 + x^2 + y^2}$ at the level $c = 2$.

(a) Analytically verify that the curve is a circle.

(b) At the point $(\sqrt{3}, 2)$ on the level curve, sketch the vector showing the direction of the greatest rate of increase of the function. (To print an enlarged copy of the graph, select the MathGraph button.)

(c) At the point $(\sqrt{3}, 2)$ on the level curve, sketch the vector such that the directional derivative is 0.

(d) Use a computer algebra system to graph the surface to verify your answers in parts (a)–(c).

In Exercises 55–58, find a normal vector to the level curve $f(x, y) = c$ at $P$.

55. $f(x, y) = x^2 + y^2$  
$c = 25, \ P(3, 4)$

56. $f(x, y) = 6 - 2x - 3y$  
$c = 6, \ P(0, 0)$

57. $f(x, y) = \frac{x}{x^2 + y^2}$  
$c = \frac{1}{2}, \ P(1, 1)$

58. $f(x, y) = xy$  
$c = -3, \ P(-1, 3)$

In Exercises 59–62, use the gradient to find a unit normal vector to the graph of the equation at the given point. Sketch your results.

59. $4x^2 - y = 6, \ (2, 10)$

60. $3x^2 - 2y^2 = 1, \ (1, 1)$

61. $9x^2 + 4y^2 = 40, \ (2, -1)$

62. $xe^y - y = 5, \ (5, 0)$

63. Temperature Distribution The temperature at the point $(x, y)$ on a metal plate is $T = \frac{x}{x^2 + y^2}$.

Find the direction of greatest increase in heat from the point $(3, 4)$.

64. Topography The surface of a mountain is modeled by the equation $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$. A mountaintop climber is at the point $(500, 300, 4390)$. In what direction should the climber move in order to ascend at the greatest rate?
Writing About Concepts

65. Define the derivative of the function \( z = f(x, y) \) in the direction \( \mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \).

66. In your own words, give a geometric description of the directional derivative of \( z = f(x, y) \).

67. Write a paragraph describing the directional derivative of the function \( f \) in the direction \( \mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \) when (a) \( \theta = 0^\circ \) and (b) \( \theta = 90^\circ \).

68. Define the gradient of a function of two variables. State the properties of the gradient.

69. Sketch the graph of a surface and select a point \( P \) on the surface. Sketch a vector in the \( xy \)-plane giving the direction of steepest ascent on the surface at \( P \).

70. Describe the relationship of the gradient to the level curves of a surface given by \( z = f(x, y) \).

71. Topography The figure shows a topographic map carried by a group of hikers. Sketch the paths of steepest descent if the hikers start at point \( A \) and if they start at point \( B \). (To print an enlarged copy of the graph, select the MathGraph button.)

72. Meteorology Meteorologists measure the atmospheric pressure in units called millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). These are level curves to the function \( P(x, y) \) yielding the pressure at any point. Sketch the gradients to the isobars at the points \( A, B, \) and \( C \). Although the magnitudes of the gradients are unknown, their lengths relative to each other can be estimated. At which of the three points is the wind speed greatest if the speed increases as the pressure gradient increases? (To print an enlarged copy of the graph, select the MathGraph button.)

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Heat-Seeking Path In Exercises 73 and 74, find the path of a heat-seeking particle placed at point \( P \) on a metal plate with temperature field \( T(x, y) \).

<table>
<thead>
<tr>
<th>Temperature Field</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>73. ( T(x, y) = 400 - 2x^2 - y^2 )</td>
<td>( P(10, 10) )</td>
</tr>
<tr>
<td>74. ( T(x, y) = 100 - x^2 - 2y^2 )</td>
<td>( P(4, 3) )</td>
</tr>
</tbody>
</table>

75. Investigation A team of oceanographers is mapping the ocean floor to assist in the recovery of a sunken ship. Using sonar, they develop the model

\[ D = 250 + 30x^2 + 50 \sin \frac{\pi y}{2}, \quad 0 \leq x \leq 2, 0 \leq y \leq 2 \]

where \( D \) is the depth in meters, and \( x \) and \( y \) are the distances in kilometers.

(a) Use a computer algebra system to graph the surface.

(b) Because the graph in part (a) is showing depth, it is not a map of the ocean floor. How could the model be changed so that the graph of the ocean floor could be obtained?

(c) What is the depth of the ship if it is located at the coordinates \( x = 1 \) and \( y = 0.5 \)?

(d) Determine the steepness of the ocean floor in the positive \( x \)-direction from the position of the ship.

(e) Determine the steepness of the ocean floor in the positive \( y \)-direction from the position of the ship.

(f) Determine the direction of the greatest rate of change of depth from the position of the ship.

76. Temperature The temperature at the point \((x, y)\) on a metal plate is modeled by

\[ T(x, y) = 400e^{-\frac{(x^2 + y^2)}{2}}, \quad x \geq 0, y \geq 0. \]

(a) Use a computer algebra system to graph the temperature distribution function.

(b) Find the directions of no change in heat on the plate from the point \((3, 5)\).

(c) Find the direction of greatest increase in heat from the point \((3, 5)\).

True or False? In Exercises 77–80, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

77. If \( f(x, y) = \sqrt{1 - x^2 - y^2} \), then \( D_u f(0, 0) = 0 \) for any unit vector \( \mathbf{u} \).

78. If \( f(x, y) = x + y \), then \(-1 \leq D_u f(x, y) \leq 1 \).

79. If \( D_u f(x, y) \) exists, then \( D_u f(x, y) = -D_{-u} f(x, y) \).

80. If \( D_u f(x_0, y_0) = c \) for any unit vector \( \mathbf{u} \), then \( c = 0 \).

81. Find a function \( f \) such that

\[ \nabla f = e^t \cos y \mathbf{i} - e^t \sin y \mathbf{j} + z \mathbf{k}. \]
**Section 13.7**

**Tangent Planes and Normal Lines**

- Find equations of tangent planes and normal lines to surfaces.
- Find the angle of inclination of a plane in space.
- Compare the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.

**Tangent Plane and Normal Line to a Surface**

So far you have represented surfaces in space primarily by equations of the form

$$z = f(x, y).$$

Equation of a surface $S$

In the development to follow, however, it is convenient to use the more general representation for a surface given by $F(x, y, z) = 0$. For a surface $S$ given by $z = f(x, y)$, you can convert to the general form by defining $F$ as

$$F(x, y, z) = f(x, y) - z.$$

Because $f(x, y) - z = 0$, you can consider $S$ to be the level surface of $F$ given by

$$F(x, y, z) = 0.$$

Alternative equation of surface $S$

**EXAMPLE 1**  **Writing an Equation of a Surface**

For the function given by

$$F(x, y, z) = x^2 + y^2 + z^2 - 4$$

describe the level surface given by $F(x, y, z) = 0$.

**Solution**  The level surface given by $F(x, y, z) = 0$ can be written as

$$x^2 + y^2 + z^2 = 4$$

which is a sphere of radius 2 whose center is at the origin.

**Try It**  **Exploration A**

You have seen many examples of the usefulness of normal lines in applications involving curves. Normal lines are equally important in analyzing surfaces and solids. For example, consider the collision of two billiard balls. When a stationary ball is struck at a point $P$ on its surface, it moves along the line of impact determined by $P$ and the center of the ball. The impact can occur in two ways. If the cue ball is moving along the line of impact, it stops dead and imparts all of its momentum to the stationary ball, as shown in Figure 13.54. If the cue ball is not moving along the line of impact, it is deflected to one side or the other and retains part of its momentum. That part of the momentum that is transferred to the stationary ball occurs along the line of impact, regardless of the direction of the cue ball, as shown in Figure 13.55. This line of impact is called the normal line to the surface of the ball at the point $P$.

**Exploration A**

Billiard Balls and Normal Lines

In each of the three figures below, the cue ball is about to strike a stationary ball at point $P$. Explain how you can use the normal line to the stationary ball at point $P$ to describe the resulting motion of each of the two balls. Assuming that each cue ball has the same speed, which stationary ball will acquire the greatest speed? Which will acquire the least? Explain your reasoning.
In the process of finding a normal line to a surface, you are also able to solve the problem of finding a **tangent plane** to the surface. Let $S$ be a surface given by

$$F(x, y, z) = 0$$

and let $P(x_0, y_0, z_0)$ be a point on $S$. Let $C$ be a curve on $S$ through $P$ that is defined by the vector-valued function

$$r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$  

Then, for all $t$,

$$F(x(t), y(t), z(t)) = 0.$$  

If $F$ is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, it follows from the Chain Rule that

$$0 = F'(t) = F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t).$$

At $(x_0, y_0, z_0)$, the equivalent vector form is

$$0 = \nabla F(x_0, y_0, z_0) \cdot r'(t).$$

This result means that the gradient at $P$ is orthogonal to the tangent vector of every curve on $S$ through $P$. So, all tangent lines on $S$ lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains $P$, as shown in Figure 13.56.

**Definition of Tangent Plane and Normal Line**

Let $F$ be differentiable at the point $P(x_0, y_0, z_0)$ on the surface $S$ given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq 0$.

1. The plane through $P$ that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to $S$ at $P$**.
2. The line through $P$ having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to $S$ at $P$**.

**NOTE** In the remainder of this section, assume $\nabla F(x_0, y_0, z_0)$ to be nonzero unless stated otherwise.

To find an equation for the tangent plane to $S$ at $(x_0, y_0, z_0)$, let $(x, y, z)$ be an arbitrary point in the tangent plane. Then the vector

$$\mathbf{v} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

lies in the tangent plane. Because $\nabla F(x_0, y_0, z_0)$ is normal to the tangent plane at $(x_0, y_0, z_0)$, it must be orthogonal to every vector in the tangent plane, and you have $\nabla F(x_0, y_0, z_0) \cdot \mathbf{v} = 0$, which leads to the following theorem.

**THEOREM 13.13 Equation of Tangent Plane**

If $F$ is differentiable at $(x_0, y_0, z_0)$, then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at $(x_0, y_0, z_0)$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$
**EXAMPLE 2** Finding an Equation of a Tangent Plane

Find an equation of the tangent plane to the hyperboloid given by

\[ z^2 - 2x^2 - 2y^2 = 12 \]

at the point \((1, -1, 4)\).

**Solution** Begin by writing the equation of the surface as

\[ z^2 - 2x^2 - 2y^2 = 12 \]

Then, considering

\[ F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12 \]

you have

\[ F_x(x, y, z) = -4x, \quad F_y(x, y, z) = -4y, \quad \text{and} \quad F_z(x, y, z) = 2z. \]

At the point \((1, -1, 4)\) the partial derivatives are

\[ F_x(1, -1, 4) = -4, \quad F_y(1, -1, 4) = 4, \quad \text{and} \quad F_z(1, -1, 4) = 8. \]

So, an equation of the tangent plane at \((1, -1, 4)\) is

\[
-4x + 4y + 4z - 24 = 0 \\
-4x + 4y + 8z - 24 = 0 \\
x - y - 2z + 6 = 0.
\]

Figure 13.57 shows a portion of the hyperboloid and tangent plane.

**TECHNOLOGY** Some three-dimensional graphing utilities are capable of graphing tangent planes to surfaces. Two examples are shown below.

To find the equation of the tangent plane at a point on a surface given by \(z = f(x, y)\), you can define the function \(F\) by

\[ F(x, y, z) = f(x, y) - z. \]

Then \(S\) is given by the level surface \(F(x, y, z) = 0\), and by Theorem 13.13 an equation of the tangent plane to \(S\) at the point \((x_0, y_0, z_0)\) is

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.
\]
**EXAMPLE 3** Finding an Equation of the Tangent Plane

Find the equation of the tangent plane to the paraboloid

\[ z = 1 - \frac{1}{10}(x^2 + 4y^2) \]

at the point \((1, 1, \frac{1}{2})\).

**Solution** From \( z = f(x, y) = 1 - \frac{1}{10}(x^2 + 4y^2) \), you obtain

\[ f_x(x, y) = -\frac{x}{5} \implies f_x(1, 1) = -\frac{1}{5} \]

and

\[ f_y(x, y) = -\frac{4y}{5} \implies f_y(1, 1) = -\frac{4}{5} \]

So, an equation of the tangent plane at \((1, 1, \frac{1}{2})\) is

\[ f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) - \left(z - \frac{1}{2}\right) = 0 \]

\[ -\frac{1}{5}(x - 1) - \frac{4}{5}(y - 1) - \left(z - \frac{1}{2}\right) = 0 \]

\[ -\frac{1}{5}x - \frac{4}{5}y - z + \frac{3}{2} = 0. \]

This tangent plane is shown in Figure 13.58.

**Try It**

The gradient \(\nabla F(x, y, z)\) gives a convenient way to find equations of normal lines, as shown in Example 4.

**EXAMPLE 4** Finding an Equation of a Normal Line to a Surface

Find a set of symmetric equations for the normal line to the surface given by \(xyz = 12\) at the point \((2, -2, -3)\).

**Solution** Begin by letting

\[ F(x, y, z) = xyz - 12. \]

Then, the gradient is given by

\[ \nabla F(x, y, z) = F_x(x, y, z)i + F_y(x, y, z)j + F_z(x, y, z)k \]

\[ = yz i + xz j + xy k \]

and at the point \((2, -2, -3)\) you have

\[ \nabla F(2, -2, -3) = (-2)(-3)i + (2)(-3)j + (2)(-2)k \]

\[ = 6i - 6j - 4k. \]

The normal line at \((2, -2, -3)\) has direction numbers 6, -6, and -4, and the corresponding set of symmetric equations is

\[ \frac{x - 2}{6} = \frac{y + 2}{-6} = \frac{z + 3}{-4}. \]

This tangent plane is shown in Figure 13.58.

**Try It**
Knowing that the gradient \( \nabla F(x, y, z) \) is normal to the surface given by \( F(x, y, z) = 0 \) allows you to solve a variety of problems dealing with surfaces and curves in space.

**EXAMPLE 5** Finding the Equation of a Tangent Line to a Curve

Describe the tangent line to the curve of intersection of the surfaces

\[
\begin{align*}
\text{Ellipsoid:} & \quad x^2 + 2y^2 + 2z^2 = 20 \\
\text{Paraboloid:} & \quad x^2 + y^2 + z = 4
\end{align*}
\]

at the point \((0, 1, 3)\), as shown in Figure 13.60.

**Solution** Begin by finding the gradients to both surfaces at the point \((0, 1, 3)\).

\[
\begin{align*}
\nabla F(x, y, z) & = 2x \mathbf{i} + 4y \mathbf{j} + 4z \mathbf{k} \\
\nabla G(x, y, z) & = 2x \mathbf{i} + 2y \mathbf{j} + \mathbf{k}
\end{align*}
\]

The cross product of these two gradients is a vector that is tangent to both surfaces at the point \((0, 1, 3)\).

\[
\nabla F(0, 1, 3) \times \nabla G(0, 1, 3) = \begin{vmatrix} 
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 4 & 12 \\
0 & 2 & 1 
\end{vmatrix} = -20\mathbf{i}.
\]

So, the tangent line to the curve of intersection of the two surfaces at the point \((0, 1, 3)\) is a line that is parallel to the \(x\)-axis and passes through the point \((0, 1, 3)\).

**The Angle of Inclination of a Plane**

Another use of the gradient \( \nabla F(x, y, z) \) is to determine the angle of inclination of the tangent plane to a surface. The **angle of inclination** of a plane is defined to be the angle \( \theta \) \((0 \leq \theta \leq \pi/2)\) between the given plane and the \(xy\)-plane, as shown in Figure 13.61. (The angle of inclination of a horizontal plane is defined to be zero.) Because the vector \( \mathbf{k} \) is normal to the \(xy\)-plane, you can use the formula for the cosine of the angle between two planes (given in Section 11.5) to conclude that the angle of inclination of a plane with normal vector \( \mathbf{n} \) is given by

\[
\cos \theta = \frac{\mathbf{n} \cdot \mathbf{k}}{||\mathbf{n}||} = \frac{\mathbf{n} \cdot \mathbf{k}}{||\mathbf{n}||}.
\]

The angle of inclination

Figure 13.61
**EXAMPLE 6** Finding the Angle of Inclination of a Tangent Plane

Find the angle of inclination of the tangent plane to the ellipsoid given by

\[
\frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} = 1
\]

at the point \((2, 2, 1)\).

**Solution** If you let

\[
F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} - 1
\]

the gradient of \(F\) at the point \((2, 2, 1)\) is given by

\[
\nabla F(x, y, z) = \frac{x}{6}i + \frac{y}{6}j + \frac{2z}{3}k
\]

\[
\nabla F(2, 2, 1) = \frac{1}{3}i + \frac{1}{3}j + \frac{2}{3}k.
\]

Because \(\nabla F(2, 2, 1)\) is normal to the tangent plane and \(k\) is normal to the \(xy\)-plane, it follows that the angle of inclination of the tangent plane is given by

\[
\cos \theta = \frac{|\nabla F(2, 2, 1) \cdot k|}{\|\nabla F(2, 2, 1)\|} = \frac{2/3}{\sqrt{(1/3)^2 + (1/3)^2 + (2/3)^2}} = \frac{\sqrt{2}}{3}
\]

which implies that

\[
\theta = \arccos \left(\frac{\sqrt{2}}{3}\right) = 35.3^\circ,
\]

as shown in Figure 13.62.

**Try It**

**Exploration A**

NOTE A special case of the procedure shown in Example 6 is worth noting. The angle of inclination \(\theta\) of the tangent plane to the surface \(z = f(x, y)\) at \((x_0, y_0, z_0)\) is given by

\[
\cos \theta = \frac{1}{\sqrt{[f(x_0, y_0)]^2 + [f(x_0, y_0)]^2 + 1}}.
\]

**A Comparison of the Gradients \(\nabla f(x, y)\) and \(\nabla f(x, y, z)\)**

This section concludes with a comparison of the gradients \(\nabla f(x, y)\) and \(\nabla f(x, y, z)\). In the preceding section, you saw that the gradient of a function \(f\) of two variables is normal to the level curves of \(f\). Specifically, Theorem 13.12 states that if \(f\) is differentiable at \((x_0, y_0)\) and \(\nabla f(x_0, y_0) \neq 0\), then \(\nabla f(x_0, y_0)\) is normal to the level curve through \((x_0, y_0)\). Having developed normal lines to surfaces, you can now extend this result to a function of three variables. The proof of Theorem 13.14 is left as an exercise (see Exercise 63).

**THEOREM 13.14 Gradient Is Normal to Level Surfaces**

If \(F\) is differentiable at \((x_0, y_0, z_0)\) and \(\nabla F(x_0, y_0, z_0) \neq 0\), then \(\nabla F(x_0, y_0, z_0)\) is normal to the level surface through \((x_0, y_0, z_0)\).

When working with the gradients \(\nabla f(x, y)\) and \(\nabla f(x, y, z)\), be sure you remember that \(\nabla f(x, y)\) is a vector in the \(xy\)-plane and \(\nabla f(x, y, z)\) is a vector in space.
Exercises for Section 13.7

The symbol \( \boxed{\text{H}} \) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on \( \boxed{G} \) to view the complete solution of the exercise.

Click on \( \boxed{R} \) to print an enlarged copy of the graph.

In Exercises 1–4, describe the level surface \( F(x, y, z) = 0 \).

1. \( F(x, y, z) = 3x - 5y + 3z - 15 \)
2. \( F(x, y, z) = x^2 + y^2 + z^2 - 25 \)
3. \( F(x, y, z) = 4x^2 + 9y^2 - 4z^2 \)
4. \( F(x, y, z) = 16x^2 - 9y^2 + 144z \)

In Exercises 5–14, find a unit normal vector to the surface at the given point. [Hint: Normalize the gradient vector \( \nabla F(x, y, z) \).

<table>
<thead>
<tr>
<th>Surface</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. ( x + y + z = 4 )</td>
<td>(2, 0, 2)</td>
</tr>
<tr>
<td>6. ( x^2 + y^2 + z^2 = 11 )</td>
<td>(3, 1, 1)</td>
</tr>
<tr>
<td>7. ( z = \sqrt{x^2 + y^2} )</td>
<td>(3, 4, 5)</td>
</tr>
<tr>
<td>8. ( z = x^3 )</td>
<td>(2, 1, 8)</td>
</tr>
<tr>
<td>9. ( x^2y - z = 0 )</td>
<td>(1, 2, 16)</td>
</tr>
<tr>
<td>10. ( x^2 + 3y + z^3 = 9 )</td>
<td>(2, -1, 2)</td>
</tr>
<tr>
<td>11. ( \ln\left(\frac{x}{y - z}\right) = 0 )</td>
<td>(1, 4, 3)</td>
</tr>
<tr>
<td>12. ( ze^{x - y} - 3 = 0 )</td>
<td>(2, 2, 3)</td>
</tr>
<tr>
<td>13. ( z - x \sin y = 4 )</td>
<td>( \left(\frac{\pi}{3}, \frac{\pi}{6}, -\frac{3}{2}\right) )</td>
</tr>
<tr>
<td>14. ( \sin(x - y) - z = 2 )</td>
<td>( \left(\frac{\pi}{3}, \frac{\pi}{6}, -\frac{3}{2}\right) )</td>
</tr>
</tbody>
</table>

In Exercises 15–18, find an equation of the tangent plane to the surface at the given point.

15. \( z = 25 - x^2 - y^2 \) at \( (3, 1, 15) \)
16. \( f(x, y) = \frac{y}{x} \) at \( (1, 2, 2) \)

17. \( z = \sqrt{x^2 + y^2} \) at \( (3, 4, 5) \)

18. \( g(x, y) = \arctan\left(\frac{y}{x}\right) \) at \( (1, 0, 0) \)

In Exercises 19–28, find an equation of the tangent plane to the surface at the given point.

19. \( g(x, y) = x^2 - y^2 \) at \( (5, 4, 9) \)
20. \( f(x, y) = 2 - \frac{1}{2}x - y \) at \( (3, -1, 1) \)
21. \( z = e^y \sin y + 1 \) at \( \left(0, \frac{\pi}{2}, 2\right) \)
22. \( z = x^2 - 2xy + y^2 \) at \( (1, 2, 1) \)
23. \( h(x, y) = \ln\sqrt{x^2 + y^2} \) at \( (3, 4, \ln 5) \)
24. \( h(x, y) = \cos y \) at \( \left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) \)
25. \( x^2 + 4y^2 + z^2 = 36 \) at \( (2, -2, 4) \)
26. \( x^2 + 2z^2 = y^2 \) at \( (1, 3, -2) \)
27. \( xy^2 + 3x - z^2 = 4 \) at \( (2, 1, -2) \)
28. \( x = y(2z - 3) \) at \( (4, 4, 2) \)

In Exercises 29–34, find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

29. \( x^2 + y^2 + z = 9 \) at \( (1, 2, 4) \)
30. \( x^2 + y^2 + z^2 = 9 \) at \( (1, 2, 2) \)
31. \( xy - z = 0 \) at \( (-2, -3, 6) \)
32. \( x^2 - y^2 + z^2 = 0 \) at \( (5, 13, -12) \)
33. \( z = \arctan\left(\frac{y}{x}\right) \) at \( (1, 1, \frac{\pi}{4}) \)
34. \( xyz = 10 \) at \( (1, 2, 5) \)
35. **Investigation** Consider the function
\[ f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)} \]
on the intervals \(-2 \leq x \leq 2\) and \(0 \leq y \leq 3\).
(a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point \((1, 1, 1)\).
(b) Repeat part (a) for the point \((-1, 2, \frac{-3}{2})\).
(c) Use a computer algebra system to graph the surface, the normal lines, and the tangent lines found in parts (a) and (b).
(d) Use analytic and graphical analysis to write a brief description of the surface at the two indicated points.

36. **Investigation** Consider the function
\[ f(x, y) = \frac{\sin y}{x} \]
on the intervals \(-3 \leq x \leq 3\) and \(0 \leq y \leq 2\pi\).
(a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point \((2, \frac{\pi}{2}, \frac{1}{2})\).
(b) Repeat part (a) for the point \((\frac{2}{3}, \frac{3\pi}{2}, \frac{3}{2})\).
(c) Use a computer algebra system to graph the surface, the normal lines, and the tangent lines found in parts (a) and (b).
(d) Use analytic and graphical analysis to write a brief description of the surface at the two indicated points.

### Writing About Concepts

37. Consider the function \(F(x, y, z) = 0\), which is differentiable at \((x_0, y_0, z_0)\). Give the definition of the tangent plane at \(P\) and the normal line at \(P\).
38. Give the standard form of the equation of the tangent plane to a surface given by \(F(x, y, z) = 0\) at \((x_0, y_0, z_0)\).
39. For some surfaces, the normal lines at any point pass through the same geometric object. What is the common geometric object for a sphere? What is the common geometric object for a right circular cylinder? Explain.
40. Discuss the relationship between the tangent plane to a surface and approximation by differentials.

In Exercises 41–46, (a) find symmetric equations of the tangent line to the curve of intersection of the surfaces at the given point, and (b) find the cosine of the angle between the gradient vectors at this point. State whether or not the surfaces are orthogonal at the point of intersection.

41. \(x^2 + y^2 = 5\), \(z = x\), \((2, 1, 2)\)
42. \(z = x^2 + y^2\), \(z = 4 - y\), \((2, -1, 5)\)
43. \(x^2 + z^2 = 25\), \(y^2 + z^2 = 25\), \((3, 3, 4)\)
44. \(z = \sqrt{x^2 + y^2}\), \(5x - 2y + 3z = 22\), \((3, 4, 5)\)
45. \(x^2 + y^2 + z^2 = 6\), \(x - y - z = 0\), \((2, 1, 1)\)
46. \(z = x^2 + y^2\), \(x + y + 6z = 33\), \((1, 2, 5)\)

47. Consider the functions
\[ f(x, y) = 6 - x^2 - y^2/4 \quad \text{and} \quad g(x, y) = 2x + y. \]
(a) Find a set of parametric equations of the tangent line to the curve of intersection of the surfaces at the point \((1, 2, 4)\) and find the angle between the gradient vectors.
(b) Use a computer algebra system to graph the surfaces.

48. Consider the functions
\[ f(x, y) = \sqrt{16 - x^2 - y^2} + 2x - 4y \]
and
\[ g(x, y) = \sqrt{\frac{9}{2}} \sqrt{1 - 3x^2 + y^2 + 2ax + 4y}. \]
(a) Use a computer algebra system to graph the first-octant portion of the surfaces represented by \(f\) and \(g\).
(b) Find one first-octant point on the curve of intersection and show that the surfaces are orthogonal at this point.
(c) These surfaces are orthogonal along the curve of intersection. Does part (b) prove this fact? Explain.

In Exercises 49–52, find the angle of inclination \(\theta\) of the tangent plane to the surface at the given point.

49. \(3x^2 + 2y^2 - z = 15\), \((2, 2, 5)\)
50. \(2xy - z^3 = 0\), \((2, 2, 2)\)
51. \(x^2 - y^2 + z = 0\), \((1, 2, 3)\)
52. \(x^2 + y^2 = 5\), \((2, 1, 3)\)

In Exercises 53 and 54, find the point on the surface where the tangent plane is horizontal. Use a computer algebra system to graph the surface and the horizontal tangent plane. Describe the surface where the tangent plane is horizontal.

53. \(z = 3 - x^2 - y^2 + 6y\)
54. \(z = 3x^2 + 2y^2 - 3x + 4y - 5\)

**Heat-Seeking Path** In Exercises 55 and 56, find the path of a heat-seeking particle placed at the given point in space with a temperature field \(T(x, y, z)\).

55. \(T(x, y, z) = 400 - 2x^2 - y^2 - 4z^2\), \((4, 3, 10)\)
56. \(T(x, y, z) = 100 - 3x - y - z^2\), \((2, 2, 5)\)

In Exercises 57 and 58, show that the tangent plane to the quadric surface at the point \((x_0, y_0, z_0)\) can be written in the given form.

57. Ellipsoid: \(x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\)
\[
\text{Plane: } \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1
\]
58. Hyperboloid: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \)

Plane: \( \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} = 1 \)

59. Show that any tangent plane to the cone

\[ z^2 = a^2x^2 + b^2y^2 \]

passes through the origin.

60. Let \( f \) be a differentiable function and consider the surface \( z = xf(y/x) \). Show that the tangent plane at any point \( P(x_0, y_0, z_0) \) on the surface passes through the origin.

61. **Approximation** Consider the following approximations for a function \( f(x, y) \) centered at \((0, 0)\).

   **Linear approximation:**
   \[ P_1(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y \]

   **Quadratic approximation:**
   \[ P_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2 \]

   [Note that the linear approximation is the tangent plane to the surface at \((0, 0, f(0, 0))\).]

   (a) Find the linear approximation of \( f(x, y) = e^{x-y} \) centered at \((0, 0)\).

   (b) Find the quadratic approximation of \( f(x, y) = e^{x-y} \) centered at \((0, 0)\).

   (c) If \( x = 0 \) in the quadratic approximation, you obtain the second-degree Taylor polynomial for what function? Answer the same question for \( y = 0 \).

   (d) Complete the table.

   \[
   \begin{array}{|c|c|c|c|}
   \hline
   x & y & f(x, y) & P_1(x, y) & P_2(x, y) \\
   \hline
   0 & 0 & & & \\
   0 & 0.1 & & & \\
   0.2 & 0.1 & & & \\
   0.2 & 0.5 & & & \\
   1 & 0.5 & & & \\
   \hline
   \end{array}
   \]

   (e) Use a computer algebra system to graph the surfaces \( z = f(x, y) \), \( z = P_1(x, y) \), and \( z = P_2(x, y) \).

62. **Approximation** Repeat Exercise 61 for the function \( f(x, y) = \cos(x + y) \).


64. Prove that the angle of inclination \( \theta \) of the tangent plane to the surface \( z = f(x, y) \) at the point \((x_0, y_0, z_0)\) is given by

\[
\cos \theta = \frac{1}{\sqrt{\left[f_x(x_0, y_0)\right]^2 + \left[f_y(x_0, y_0)\right]^2 + 1}}
\]
### Extrema of Functions of Two Variables

- Find absolute and relative extrema of a function of two variables.
- Use the Second Partials Test to find relative extrema of a function of two variables.

#### Absolute Extrema and Relative Extrema

In Chapter 3, you studied techniques for finding the extreme values of a function of a single variable. In this section, you will extend these techniques to functions of two variables. For example, in Theorem 13.15 the Extreme Value Theorem for a function of a single variable is extended to a function of two variables.

Consider the continuous function of two variables, defined on a closed bounded region $R$. The values $f(a, b)$ and $f(c, d)$ such that

$$f(a, b) \leq f(x, y) \leq f(c, d)$$

for all $(x, y)$ in $R$ are called the **minimum** and **maximum** of $f$ in the region $R$, as shown in Figure 13.63. Recall from Section 13.2 that a region in the plane is **closed** if it contains all of its boundary points. The Extreme Value Theorem deals with a region in the plane that is both closed and **bounded**. A region in the plane is called **bounded** if it is a subregion of a closed disk in the plane.

#### THEOREM 13.15 Extreme Value Theorem

Let $f$ be a continuous function of two variables $x$ and $y$ defined on a closed bounded region $R$ in the $xy$-plane.

1. There is at least one point in $R$ where $f$ takes on a minimum value.
2. There is at least one point in $R$ where $f$ takes on a maximum value.

A minimum is also called an **absolute minimum** and a maximum is also called an **absolute maximum**. As in single-variable calculus, there is a distinction made between absolute extrema and relative extrema.

#### Definition of Relative Extrema

Let $f$ be a function defined on a region $R$ containing $(x_0, y_0)$.

1. The function $f$ has a **relative minimum** at $(x_0, y_0)$ if

   $$f(x, y) \geq f(x_0, y_0)$$

   for all $(x, y)$ in an open disk containing $(x_0, y_0)$.

2. The function $f$ has a **relative maximum** at $(x_0, y_0)$ if

   $$f(x, y) \leq f(x_0, y_0)$$

   for all $(x, y)$ in an open disk containing $(x_0, y_0)$.

To say that $f$ has a relative maximum at $(x_0, y_0)$ means that the point $(x_0, y_0, z_0)$ is at least as high as all nearby points on the graph of $z = f(x, y)$. Similarly, $f$ has a relative minimum at $(x_0, y_0)$ if $(x_0, y_0, z_0)$ is at least as low as all nearby points on the graph. (See Figure 13.64.)
To locate relative extrema of $f$, you can investigate the points at which
the gradient of $f$ is $0$ or the points at which one of the partial derivatives does not exist.
Such points are called critical points of $f$.

**Definition of Critical Point**

Let $f$ be defined on an open region $R$ containing $(x_0, y_0)$. The point $(x_0, y_0)$ is a
critical point of $f$ if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Recall from Theorem 13.11 that if $f$ is differentiable and

$$
\nabla f(x_0, y_0) = f_x(x_0, y_0)i + f_y(x_0, y_0)j
= 0i + 0j
$$

then every directional derivative at $(x_0, y_0)$ must be $0$. This implies that the function
has a horizontal tangent plane at the point $(x_0, y_0)$, as shown in Figure 13.65. It appears
that such a point is a likely location of a relative extremum. This is confirmed by
Theorem 13.16.

**THEOREM 13.16 Relative Extrema Occur Only at Critical Points**

If $f$ has a relative extremum at $(x_0, y_0)$ on an open region $R$, then $(x_0, y_0)$ is a
critical point of $f$.

**EXPLORATION**

Use a graphing utility to graph

$$
z = x^3 - 3xy + y^3
$$

using the bounds $0 \leq x \leq 3$, $0 \leq y \leq 3$, and $-3 \leq z \leq 3$. This view makes it appear as though
the surface has an absolute minimum. But does it?
EXAMPLE 1 Finding a Relative Extremum

Determine the relative extrema of
\[ f(x, y) = 2x^2 + y^2 + 8x - 6y + 20. \]

Solution Begin by finding the critical points of \( f \). Because
\[ f_x(x, y) = 4x + 8 \quad \text{Partial with respect to } x \]
and
\[ f_y(x, y) = 2y - 6 \quad \text{Partial with respect to } y \]
are defined for all \( x \) and \( y \), the only critical points are those for which both first partial derivatives are 0. To locate these points, let \( f_x(x, y) \) and \( f_y(x, y) \) be 0, and solve the equations
\[ 4x + 8 = 0 \quad \text{and} \quad 2y - 6 = 0 \]
to obtain the critical point \((-2, 3)\). By completing the square, you can conclude that for all \((x, y) \neq (-2, 3)\)
\[ f(x, y) = 2(x + 2)^2 + (y - 3)^2 + 3 > 3. \]
So, a relative minimum of \( f \) occurs at \((-2, 3)\). The value of the relative minimum is \( f(-2, 3) = 3 \), as shown in Figure 13.66.

Example 1 shows a relative minimum occurring at one type of critical point—the type for which both \( f_x(x, y) \) and \( f_y(x, y) \) are 0. The next example concerns a relative maximum that occurs at the other type of critical point—the type for which either \( f_x(x, y) \) or \( f_y(x, y) \) does not exist.

Try It Exploration A Exploration B Open Exploration

EXAMPLE 2 Finding a Relative Extremum

Determine the relative extrema of \( f(x, y) = 1 - (x^2 + y^2)^{1/3} \).

Solution Because
\[ f_x(x, y) = -\frac{2x}{3(x^2 + y^2)^{2/3}} \quad \text{Partial with respect to } x \]
and
\[ f_y(x, y) = -\frac{2y}{3(x^2 + y^2)^{2/3}} \quad \text{Partial with respect to } y \]
it follows that both partial derivatives exist for all points in the \( xy \)-plane except for \((0, 0)\). Moreover, because the partial derivatives cannot both be 0 unless both \( x \) and \( y \) are 0, you can conclude that \((0, 0)\) is the only critical point. In Figure 13.67, note that \( f(0, 0) \) is 1. For all other \((x, y)\) it is clear that
\[ f(x, y) = 1 - (x^2 + y^2)^{1/3} < 1. \]
So, \( f \) has a relative maximum at \((0, 0)\).

NOTE In Example 2, \( f_x(x, y) = 0 \) for every point on the \( y \)-axis other than \((0, 0)\). However, because \( f_y(x, y) \) is nonzero, these are not critical points. Remember that one of the partials must not exist or both must be 0 in order to yield a critical point.
The Second Partials Test

Theorem 13.16 tells you that to find relative extrema you need only examine values of \( f(x, y) \) at critical points. However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima. Some critical points yield saddle points, which are neither relative maxima nor relative minima.

As an example of a critical point that does not yield a relative extremum, consider the surface given by

\[
f(x, y) = y^2 - x^2
\]

as shown in Figure 13.68. At the point \((0, 0)\), both partial derivatives are 0. The function \(f\) does not, however, have a relative extremum at this point because in any open disk centered at \((0, 0)\) the function takes on both negative values (along the \(x\)-axis) and positive values (along the \(y\)-axis). So, the point \((0, 0, 0)\) is a saddle point of the surface. (The term “saddle point” comes from the fact that the surface shown in Figure 13.68 resembles a saddle.)

For the functions in Examples 1 and 2, it was relatively easy to determine the relative extrema, because each function was either given, or able to be written, in completed square form. For more complicated functions, algebraic arguments are less convenient and it is better to rely on the analytic means presented in the following Second Partials Test. This is the two-variable counterpart of the Second Derivative Test for functions of one variable. The proof of this theorem is best left to a course in advanced calculus.

**THEOREM 13.17 Second Partials Test**

Let \( f \) have continuous second partial derivatives on an open region containing a point \((a, b)\) for which

\[
f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.
\]

To test for relative extrema of \( f \), consider the quantity

\[
d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.
\]

1. If \( d > 0 \) and \( f_{xx}(a, b) > 0 \), then \( f \) has a relative minimum at \((a, b)\).
2. If \( d > 0 \) and \( f_{xx}(a, b) < 0 \), then \( f \) has a relative maximum at \((a, b)\).
3. If \( d < 0 \), then \((a, b, f(a, b))\) is a saddle point.
4. The test is inconclusive if \( d = 0 \).

NOTE If \( d > 0 \), then \( f_{xx}(a, b) \) and \( f_{yy}(a, b) \) must have the same sign. This means that \( f_{xx}(a, b) \) can be replaced by \( f_{yy}(a, b) \) in the first two parts of the test.

A convenient device for remembering the formula for \( d \) in the Second Partials Test is given by the \(2 \times 2\) determinant

\[
d = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}
\]

where \( f_{yx}(a, b) = f_{xy}(a, b) \) by Theorem 13.3.
**Example 3 Using the Second Partial\s Test**

Find the relative extrema of \( f(x, y) = -x^3 + 4xy - 2y^2 + 1 \).

**Solution** Begin by finding the critical points of \( f \). Because 

\[
  f_x(x, y) = -3x^2 + 4y \quad \text{and} \quad f_y(x, y) = 4x - 4y
\]

exist for all \( x \) and \( y \), the only critical points are those for which both first partial derivatives are 0. To locate these points, let \( f_x(x, y) \) and \( f_y(x, y) \) be 0 to obtain

\[
  -3x^2 + 4y = 0 \quad \text{and} \quad 4x - 4y = 0.
\]

From the second equation you know that \( x = y \), and, by substitution into the first equation, you obtain two solutions: \( y = x = 0 \) and \( y = x = \frac{2}{3} \). Because

\[
  f_{xx}(x, y) = -6x, \quad f_{yy}(x, y) = -4, \quad \text{and} \quad f_{xy}(x, y) = 4
\]

it follows that, for the critical point \((0, 0)\),

\[
  d = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 16 < 0
\]

and, by the Second Partial\s Test, you can conclude that \((0, 0, 1)\) is a saddle point of \( f \). Furthermore, for the critical point \((\frac{2}{3}, \frac{4}{3})\),

\[
  d = f_{xx}(\frac{2}{3}, \frac{4}{3})f_{yy}(\frac{2}{3}, \frac{4}{3}) - [f_{xy}(\frac{2}{3}, \frac{4}{3})]^2
  = -8(-4) - 16
  = 16
  > 0
\]

and because \( f_{xx}(\frac{2}{3}, \frac{4}{3}) = -8 < 0 \) you can conclude that \( f \) has a relative maximum at \((\frac{2}{3}, \frac{4}{3})\), as shown in Figure 13.69.

**Try It**

**Exploration A**

The Second Partial\s Test can fail to find relative extrema in two ways. If either of the first partial\s derivatives does not exist, you cannot use the test. Also, if

\[
  d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0
\]

the test fails. In such cases, you can try a sketch or some other approach, as demonstrated in the next example.

**Example 4 Failure of the Second Partial\s Test**

Find the relative extrema of \( f(x, y) = x^2y^2 \).

**Solution** Because \( f_x(x, y) = 2xy^2 \) and \( f_y(x, y) = 2x^2y \), you know that both partial\s derivatives are 0 if \( x = 0 \) or \( y = 0 \). That is, every point along the \( x- \) or \( y\)-axis is a critical point. Moreover, because

\[
  f_{xx}(x, y) = 2y^2, \quad f_{yy}(x, y) = 2x^2, \quad \text{and} \quad f_{xy}(x, y) = 4xy
\]

you know that if either \( x = 0 \) or \( y = 0 \), then

\[
  d = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2
  = 4x^2y^2 - 16x^2y^2
  = -12x^2y^2 = 0.
\]

So, the Second Partial\s Test fails. However, because \( f(x, y) = 0 \) for every point along the \( x- \) or \( y\)-axis and \( f(x, y) = x^2y^2 > 0 \) for all other points, you can conclude that each of these critical points yields an absolute minimum, as shown in Figure 13.70.

**Try It**

**Exploration A**
Absolute extrema of a function can occur in two ways. First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, \( f(-2, 3) \) is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.) Second, absolute extrema can occur at a boundary point of the domain. This is illustrated in Example 5.

**EXAMPLE 5  Finding Absolute Extrema**

Find the absolute extrema of the function

\[ f(x, y) = \sin xy \]

on the closed region given by \( 0 \leq x \leq \pi \) and \( 0 \leq y \leq 1 \).

**Solution** From the partial derivatives

\[ f_x(x, y) = y \cos xy \quad \text{and} \quad f_y(x, y) = x \cos xy \]

you can see that each point lying on the hyperbola given by \( xy = \pi/2 \) is a critical point. These points each yield the value

\[ f(x, y) = \sin \frac{\pi}{2} = 1 \]

which you know is the absolute maximum, as shown in Figure 13.71. The only other critical point of \( f \) lying in the given region is \((0, 0)\). It yields an absolute minimum of 0, because

\[ 0 \leq xy \leq \pi \]

implies that

\[ 0 \leq \sin xy \leq 1. \]

To locate other absolute extrema, you should consider the four boundaries of the region formed by taking traces with the vertical planes \( x = 0 \), \( x = \pi \), \( y = 0 \), and \( y = 1 \). In doing this, you will find that \( \sin xy = 0 \) at all points on the \( x \)-axis, at all points on the \( y \)-axis, and at the point \((\pi, 1)\). Each of these points yields an absolute minimum for the surface, as shown in Figure 13.71.

**Try It**

The concepts of relative extrema and critical points can be extended to functions of three or more variables. If all first partial derivatives of

\[ w = f(x_1, x_2, x_3, \ldots, x_n) \]

exist, it can be shown that a relative maximum or minimum can occur at \((x_1, x_2, x_3, \ldots, x_n)\) only if every first partial derivative is 0 at that point. This means that the critical points are obtained by solving the following system of equations.

\[ f_{x_1}(x_1, x_2, x_3, \ldots, x_n) = 0 \]
\[ f_{x_2}(x_1, x_2, x_3, \ldots, x_n) = 0 \]
\[ \vdots \]
\[ f_{x_n}(x_1, x_2, x_3, \ldots, x_n) = 0 \]

The extension of Theorem 13.17 to three or more variables is also possible, although you will not consider such an extension in this text.
In Exercises 21–28, examine the function for relative extrema.

1. \( g(x, y) = (x - 1)^2 + (y - 3)^2 \)
2. \( g(x, y) = 9 - (x - 3)^2 - (y + 2)^2 \)
3. \( f(x, y) = \sqrt{x^2 + y^2 + 1} \)
4. \( f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \)
5. \( f(x, y) = x^2 + y^2 + 2x - 6y + 6 \)
6. \( f(x, y) = -x^2 - y^2 + 4x + 8y - 11 \)

In Exercises 7–16, examine the function for relative extrema.

7. \( f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3 \)
8. \( f(x, y) = -x^2 - 5y^2 + 10x - 30y - 62 \)
9. \( f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10 \)
10. \( f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4 \)
11. \( z = 2x^2 + 3y^2 - 4x - 12y + 13 \)
12. \( z = -3x^2 - 2y^2 + 3x - 4y + 5 \)
13. \( f(x, y) = 2\sqrt{x^2 + y^2} + 3 \)
14. \( h(x, y) = (x^2 + y^2)^{1/3} + 2 \)
15. \( g(x, y) = 4 - |x| - |y| \)
16. \( f(x, y) = |x + y| - 2 \)

In Exercises 17–20, use a computer algebra system to graph the surface and locate any relative extrema and saddle points.

17. \( z = \frac{-4x}{x^2 + y^2 + 1} \)
18. \( f(x, y) = y^3 - 3xy^2 - 3y^2 - 3x^2 + 1 \)
19. \( z = (x^2 + 4y^2)e^{x^2 - y^2} \)
20. \( z = e^{xy} \)

In Exercises 21–28, examine the function for relative extrema and saddle points.

21. \( h(x, y) = x^2 - y^2 - 2x - 4y - 4 \)
22. \( g(x, y) = 120x + 120y - xy - x^2 - y^2 \)
23. \( h(x, y) = x^2 - 3xy - y^2 \)
24. \( g(x, y) = xy \)
25. \( f(x, y) = x^3 - 3xy + y^3 \)

26. \( f(x, y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1 \)
27. \( z = e^{-x} \sin y \)
28. \( z = \left( \frac{1}{2} - x^2 + y^2 \right)e^{1-x^2-y^2} \)

In Exercises 29 and 30, examine the function for extrema without using the derivative tests and use a computer algebra system to graph the surface. (Hint: By observation, determine if it is possible for \( z \) to be negative. When is \( z \) equal to 0?)

29. \( z = \frac{(x - y)^2}{x^2 + y^2} \)
30. \( z = \frac{(x^2 - y^2)^2}{x^3 + y^3} \)

Think About It: In Exercises 31–34, determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function \( f(x, y) \) at the critical point \( (x_0, y_0) \).
Writing About Concepts

35. Define each of the following for a function of two variables.
   (a) Relative minimum
   (b) Relative maximum
   (c) Saddle point
   (d) Critical point

36. State the Second Partials Test for relative extrema and saddle points.

In Exercises 37–40, sketch the graph of an arbitrary function $f$ satisfying the given conditions. State whether the function has any extrema or saddle points. (There are many correct answers.)

37. $f_x(x, y) > 0$ and $f_y(x, y) < 0$ for all $(x, y)$.

38. All of the first and second partial derivatives of $f$ are 0.

39. $f_x(0, 0) = 0$, $f_y(0, 0) = 0$
   
   $f_x(x, y) = \begin{cases} < 0, & x < 0 \\ > 0, & x > 0 \end{cases}$
   $f_y(x, y) = \begin{cases} < 0, & y < 0 \\ > 0, & y > 0 \end{cases}$
   
   $f_{xx}(x, y) > 0$, $f_{xy}(x, y) < 0$, and $f_{yy}(x, y) = 0$ for all $(x, y)$.

40. $f_x(2, 1) = 0$, $f_y(2, 1) = 0$
   
   $f_x(x, y) = \begin{cases} > 0, & x < 2 \\ < 0, & x > 2 \end{cases}$
   $f_y(x, y) = \begin{cases} > 0, & y < 1 \\ < 0, & y > 1 \end{cases}$
   
   $f_{xx}(x, y) < 0$, $f_{xy}(x, y) < 0$, and $f_{yy}(x, y) = 0$ for all $(x, y)$.

41. The figure shows the level curves for an unknown function $f(x, y)$. What, if any, information can be given about $f$ at the point $A$? Explain your reasoning.

42. The figure shows the level curves for an unknown function $f(x, y)$. What, if any, information can be given about $f$ at the points $A$, $B$, $C$, and $D$? Explain your reasoning.

43. A function $f$ has continuous second partial derivatives on an open region containing the critical point $(3, 7)$. The function has a minimum at $(3, 7)$ and $d > 0$ for the Second Partials Test. Determine the interval for $f_x(3, 7)$ if $f_x(3, 7) = 2$ and $f_y(3, 7) = 8$.

44. A function $f$ has continuous second partial derivatives on an open region containing the critical point $(a, b)$. If $f_x(a, b)$ and $f_y(a, b)$ have opposite signs, what is implied? Explain.

In Exercises 45–50, find the critical points and test for relative extrema. List the critical points for which the Second Partials Test fails.

45. $f(x, y) = x^3 + y^3$
46. $f(x, y) = x^4 + y^4 - 6x^2 + 9y^2 + 12x + 27y + 19$
47. $f(x, y) = (x - 1)^2(y + 4)^2$
48. $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2}$
49. $f(x, y) = x^{2/3} + y^{2/3}$
50. $f(x, y) = (x^2 + y^2)^{2/3}$

In Exercises 51 and 52, find the critical points of the function and, from the form of the function, determine whether a relative maximum or a relative minimum occurs at each point.

51. $f(x, y, z) = x^2 + (y - 1)^2 + (z + 2)^2$
52. $f(x, y, z) = 4 - [x(x - 1)(z + 2)]^2$

In Exercises 53–62, find the absolute extrema of the function over the region $R$. (In each case, $R$ contains the boundaries.) Use a computer algebra system to confirm your results.

53. $f(x, y) = 12 - 3x - 2y$
   
   $R$: The triangular region in the $xy$-plane with vertices $(2, 0)$, $(0, 1)$, and $(1, 2)$
54. $f(x, y) = (2x - y)^2$
   
   $R$: The triangular region in the $xy$-plane with vertices $(2, 0)$, $(0, 1)$, and $(1, 2)$
55. $f(x, y) = 3x^2 + 2y^2 - 4y$
   
   $R$: The region in the $xy$-plane bounded by the graphs of $y = x^2$ and $y = 4$
56. $f(x, y) = 2x - 2xy + y^2$
   
   $R$: The region in the $xy$-plane bounded by the graphs of $y = x^2$ and $y = 1$
57. $f(x, y) = x^2 + xy$, $R = \{(x, y) : |x| \leq 2, |y| \leq 1\}$
58. $f(x, y) = x^2 + 2xy + y^2$, $R = \{(x, y) : |x| \leq 2, |y| \leq 1\}$
59. $f(x, y) = x^2 + 2xy + y^2$, $R = \{(x, y) : x^2 + y^2 \leq 8\}$
60. $f(x, y) = x^2 - 4xy + 5$
   
   $R = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq \sqrt{x}\}$
61. $f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}$
   
   $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$
62. $f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}$
   
   $R = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$

True or False? In Exercises 63 and 64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

63. If $f$ has a relative maximum at $(x_0, y_0, z_0)$, then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.
64. If $f$ is continuous for all $x$ and $y$ and has two relative minima then $f$ must have at least one relative maximum.
Applications of Extrema of Functions of Two Variables

- Solve optimization problems involving functions of several variables.
- Use the method of least squares.

Applied Optimization Problems

In this section, you will survey a few of the many applications of extrema of functions of two (or more) variables.

**EXAMPLE 1** Finding Maximum Volume

A rectangular box is resting on the xy-plane with one vertex at the origin. The opposite vertex lies in the plane

\[ 6x + 4y + 3z = 24 \]

as shown in Figure 13.72. Find the maximum volume of such a box.

**Solution** Let \( x, y, \) and \( z \) represent the length, width, and height of the box. Because one vertex of the box lies in the plane \( 6x + 4y + 3z = 24 \), you know that \( z = \frac{1}{3}(24 - 6x - 4y) \), and you can write the volume \( xyz \) of the box as a function of two variables.

\[
V(x, y) = (x)(y)\left[\frac{1}{3}(24 - 6x - 4y)\right] = \frac{1}{3}(24xy - 6x^2y - 4xy^2)
\]

By setting the first partial derivatives equal to 0

\[
V_x(x, y) = \frac{1}{3}(24y - 12xy - 4y^2) = \frac{y}{3}(24 - 12x - 4y) = 0
\]

\[
V_y(x, y) = \frac{1}{3}(24x - 6x^2 - 8xy) = \frac{x}{3}(24 - 6x - 8y) = 0
\]

you obtain the critical points \((0,0)\) and \((\frac{4}{3}, 2)\). At \((0,0)\) the volume is 0, so that point does not yield a maximum volume. At the point \((\frac{4}{3}, 2)\), you can apply the Second Partials Test.

\[
V_{xx}(x, y) = -4y \\
V_{yy}(x, y) = \frac{-8x}{3} \\
V_{yx}(x, y) = \frac{1}{3}(24 - 12x - 8y)
\]

Because

\[
V_{xx}(\frac{4}{3}, 2)V_{yy}(\frac{4}{3}, 2) - [V_{yx}(\frac{4}{3}, 2)]^2 = (-8)(-\frac{12}{3}) - \left(\frac{8}{3}\right)^2 = \frac{64}{3} > 0
\]

and

\[
V_{xx}(\frac{4}{3}, 2) = -8 < 0
\]

you can conclude from the Second Partials Test that the maximum volume is

\[
V(\frac{4}{3}, 2) = \frac{1}{3}(24\left(\frac{4}{3}\right)^2) - 6\left(\frac{4}{3}\right)^2 - 4\left(\frac{4}{3}\right)^2
\]

\[
= \frac{64}{3} \text{ cubic units.}
\]

Note that the volume is 0 at the boundary points of the triangular domain of \( V \).
Applications of extrema in economics and business often involve more than one independent variable. For instance, a company may produce several models of one type of product. The price per unit and profit per unit are usually different for each model. Moreover, the demand for each model is often a function of the prices of the other models (as well as its own price). The next example illustrates an application involving two products.

**EXAMPLE 2 Finding the Maximum Profit**

An electronics manufacturer determines that the profit (in dollars) obtained by producing $x$ units of a DVD player and $y$ units of a DVD recorder is approximated by the model

$$P(x, y) = 8x + 10y - (0.001)(x^2 + xy + y^2) - 10,000.$$ 

Find the production level that produces a maximum profit. What is the maximum profit?

**Solution**

The partial derivatives of the profit function are

$$P_x(x, y) = 8 - (0.001)(2x + y) \quad \text{and} \quad P_y(x, y) = 10 - (0.001)(x + 2y).$$

By setting these partial derivatives equal to 0, you obtain the following system of equations.

$$8 - (0.001)(2x + y) = 0$$
$$10 - (0.001)(x + 2y) = 0$$

After simplifying, this system of linear equations can be written as

$$2x + y = 8000$$
$$x + 2y = 10,000.$$ 

Solving this system produces $x = 2000$ and $y = 4000$. The second partial derivatives of $P$ are

$$P_{xx}(2000, 4000) = -0.002$$
$$P_{xy}(2000, 4000) = -0.002$$
$$P_{yx}(2000, 4000) = -0.001.$$ 

Because $P_{xx} < 0$ and

$$P_{xx}(2000, 4000)P_{xy}(2000, 4000) - [P_{xy}(2000, 4000)]^2 =$$
$$(-0.002)^2 - (-0.001)^2 > 0$$

you can conclude that the production level of $x = 2000$ units and $y = 4000$ units yields a maximum profit. The maximum profit is

$$P(2000, 4000) = 8(2000) + 10(4000) -$$
$$\left(0.001\left[2000^2 + 2000(4000) + 4000^2\right]\right) - 10,000$$
$$= 818,000.$$ 

**NOTE** In Example 2, it was assumed that the manufacturing plant is able to produce the required number of units to yield a maximum profit. In actual practice, the production would be bounded by physical constraints. You will study such constrained optimization problems in the next section.
The Method of Least Squares

Many of the examples in this text have involved mathematical models. For instance, Example 2 involves a quadratic model for profit. There are several ways to develop such models; one is called the method of least squares.

In constructing a model to represent a particular phenomenon, the goals are simplicity and accuracy. Of course, these goals often conflict. For instance, a simple linear model for the points in Figure 13.73 is

\[ y = 1.8566x - 5.0246. \]

However, Figure 13.74 shows that by choosing the slightly more complicated quadratic model*

\[ y = 0.1996x^2 - 0.7281x + 1.3749 \]

you can achieve greater accuracy.

As a measure of how well the model \( y = f(x) \) fits the collection of points \( \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\} \) you can add the squares of the differences between the actual \( y \)-values and the values given by the model to obtain the sum of the squared errors

\[ S = \sum_{i=1}^{n} [f(x_i) - y_i]^2. \]

Graphically, \( S \) can be interpreted as the sum of the squares of the vertical distances between the graph of \( f \) and the given points in the plane, as shown in Figure 13.75. If the model is perfect, then \( S = 0 \). However, when perfection is not feasible, you can settle for a model that minimizes \( S \). For instance, the sum of the squared errors for the linear model in Figure 13.73 is \( S = 17 \). Statisticians call the linear model that minimizes \( S \) the least squares regression line. The proof that this line actually minimizes \( S \) involves the minimizing of a function of two variables.

* A method for finding the least squares quadratic model for a collection of data is described in Exercise 39.
Proof  Let represent the sum of the squared errors for the model $f(x) = ax + b$, where

$$S(a, b) = \sum_{i=1}^{n} [f(x_i) - y_i]^2 = \sum_{i=1}^{n} (ax_i + b - y_i)^2$$

where the points $(x_i, y_i)$ represent constants. Because $S$ is a function of $a$ and $b$, you can use the methods discussed in the preceding section to find the minimum value of $S$. Specifically, the first partial derivatives of $S$ are

$$S_a(a, b) = \sum_{i=1}^{n} 2x_i(ax_i + b - y_i) = 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i$$

$$S_b(a, b) = \sum_{i=1}^{n} 2(ax_i + b - y_i) = 2a\sum_{i=1}^{n} x_i + 2nb - 2\sum_{i=1}^{n} y_i,$$

By setting these two partial derivatives equal to 0, you obtain the values for $a$ and $b$ that are listed in the theorem. It is left to you to apply the Second Partial Test (see Exercise 40) to verify that these values of $a$ and $b$ yield a minimum.

If the $x$-values are symmetrically spaced about the $y$-axis, then $\sum x_i = 0$ and the formulas for $a$ and $b$ simplify to

$$a = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

and

$$b = \frac{1}{n} \sum_{i=1}^{n} y_i.$$
EXAMPLE 3  Finding the Least Squares Regression Line

Find the least squares regression line for the points \((-3, 0), (-1, 1), (0, 2),\) and \((2, 3)\).

Solution  The table shows the calculations involved in finding the least squares regression line using \(n = 4\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(xy)</th>
<th>(x^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

\[
\sum_{i=1}^{n} x_i = -2 \quad \sum_{i=1}^{n} y_i = 6 \quad \sum_{i=1}^{n} x_i y_i = 5 \quad \sum_{i=1}^{n} x_i^2 = 14
\]

Applying Theorem 13.18 produces

\[
a = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2} = \frac{4(5) - (-2)(6)}{4(14) - (-2)^2} = \frac{8}{13}
\]

and

\[
b = \frac{1}{n} \left( \sum_{i=1}^{n} y_i - a \sum_{i=1}^{n} x_i \right) = \frac{1}{4} \left[ 6 - \frac{8}{13}(-2) \right] = \frac{47}{26}
\]

The least squares regression line is \(f(x) = \frac{8}{13} x + \frac{47}{26}\), as shown in Figure 13.76.

TECHNOLOGY  Many calculators have “built-in” least squares regression programs. If your calculator has such a program, use it to duplicate the results of Example 3.
Exercises for Section 13.9

The symbol \(\square\) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on \(\text{S}\) to view the complete solution of the exercise.

Click on \(\text{M}\) to print an enlarged copy of the graph.

In Exercises 1 and 2, find the minimum distance from the point to the plane \(2x + 3y + z = 12\). (Hint: To simplify the computations, minimize the square of the distance.)

1. \((0, 0, 0)\)  
2. \((1, 2, 3)\)

In Exercises 3 and 4, find the minimum distance from the point to the paraboloid \(z = x^2 + y^2\).

3. \((5, 5, 0)\)  
4. \((5, 0, 0)\)

In Exercises 5–8, find three positive numbers \(x\), \(y\), and \(z\) that satisfy the given conditions.

5. The sum is 30 and the product is a maximum.
6. The sum is 32 and \(P = xyz\) is a maximum.
7. The sum is 30 and the sum of the squares is a minimum.
8. The sum is 1 and the sum of the squares is a minimum.

9. Maximum Volume The sum of the length and the girth (perimeter of a cross section) of a package carried by a delivery service cannot exceed 108 inches. Find the dimensions of the rectangular package of largest volume that may be sent.

10. Maximum Volume The material for constructing the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. For a fixed amount of money \(C\), find the dimensions of the box of largest volume that can be made.

11. Maximum Volume The volume of an ellipsoid

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

is \(4\pi abc/3\). For a fixed sum \(a + b + c\), show that the ellipsoid of maximum volume is a sphere.

12. Maximum Volume Show that the rectangular box of maximum volume inscribed in a sphere of radius \(r\) is a cube.
13. **Volume and Surface Area**  Show that a rectangular box of
given volume and minimum surface area is a cube.

14. **Maximum Volume**  Repeat Exercise 9 under the condition
that the sum of the perimeters of the two cross sections shown
in the figure cannot exceed 144 inches.

**Rotatable Graph**

15. **Area**  A trough with trapezoidal cross sections is formed by
turning up the edges of a 30-inch-wide sheet of aluminum (see
figure). Find the cross section of maximum area.

16. **Area**  Repeat Exercise 15 for a sheet that is w inches wide.

17. **Maximum Revenue**  A company manufactures two types of
sneakers, running shoes and basketball shoes. The total revenue
from \( x_1 \) units of running shoes and \( x_2 \) units of basketball shoes
is \( R = -5x_1^2 - 8x_1^2 - 2x_1x_2 + 42x_1 + 102x_2 \), where \( x_1 \) and
\( x_2 \) are in thousands of units. Find \( x_1 \) and \( x_2 \) so as to maximize
the revenue.

18. **Maximum Revenue**  A retail outlet sells two types of riding
lawn mowers, the prices of which are \( p_1 \) and \( p_2 \). Find \( p_1 \)
and \( p_2 \) so as to maximize total revenue, where
\( R = 515p_1 + 805p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2 \).

19. **Maximum Profit**  A corporation manufactures candles at two
locations. The cost of producing \( x_1 \) units at location 1 is
\[ C_1 = 0.02x_1^2 + 4x_1 + 500 \]
and the cost of producing \( x_2 \) units at location 2 is
\[ C_2 = 0.05x_2^2 + 4x_2 + 275. \]
The candles sell for $15 per unit. Find the quantity that should
be produced at each location to maximize the profit
\( P = 15(x_1 + x_2) - C_1 - C_2 \).

20. **Hardy-Weinberg Law**  Common blood types are determined
genetically by three alleles A, B, and O. (An allele is any of a
group of possible mutational forms of a gene.) A person whose
blood type is AA, BB, or OO is homozygous. A person whose
blood type is AB, AO, or BO is heterozygous. The Hardy-
Weinberg Law states that the proportion \( P \) of heterozygous
individuals in any given population is
\[ P(p, q, r) = 2pq + 2pr + 2qr \]
where \( p \) represents the percent of allele A in the population, \( q \)
represents the percent of allele B in the population, and \( r \)
represents the percent of allele O in the population. Use the fact
that \( p + q + r = 1 \) to show that the maximum proportion of
heterozygous individuals in any population is \( \frac{3}{4} \).

21. **Minimum Cost**  A water line is to be built from point \( P \) to
point \( S \) and must pass through regions where construction costs
differ (see figure). The cost per kilometer in dollars is \( \frac{3k}{2} \) from
\( P \) to \( O \), \( 2k \) from \( O \) to \( R \), and \( k \) from \( R \) to \( S \). Find \( x \) and \( y \) such
that the total cost \( C \) will be minimized.

![Image](image.png)

22. **Distance**  A company has retail outlets located at the points
\((0, 0)\), \((2, 2)\), and \((-2, 2)\) (see figure). Management plans to
build a distribution center located such that the sum of the
distances \( S \) from the center to the outlets is minimum. From the
symmetry of the problem it is clear that the distribution center
will be located on the y-axis, and therefore \( S \) is a function of the
single variable \( y \). Using techniques presented in Chapter 3, find
the required value of \( y \).

![Image](image.png)

23. **Investigation**  The retail outlets described in Exercise 22 are
located at \((0, 0)\), \((4, 2)\), and \((-2, 2)\) (see figure). The location of the
distribution center is \((x, y)\), and therefore the sum of the
distances \( S \) is a function of \( x \) and \( y \).

(a) Write the expression giving the sum of the distances \( S \). Use
a computer algebra system to graph \( S \). Does the surface
have a minimum?

(b) Use a computer algebra system to obtain \( S_x \) and \( S_y \). Observe
that solving the system \( S_x = 0 \) and \( S_y = 0 \) is very difficult.
So, approximate the location of the distribution center.

(c) An initial estimate of the critical point is \((x_1, y_1) = (1, 1)\)
Calculate \(-\nabla S(1, 1)\) with components \(-S_x(1, 1)\) and
\(-S_y(1, 1)\). What direction is given by the vector \(-\nabla S(1, 1)\)?

(d) The second estimate of the critical point is
\((x_2, y_2) = (x_1 - S_x(x_1, y_1), y_1 - S_y(x_1, y_1))\).
If these coordinates are substituted into \( S(x, y) \), then \( S \)
becomes a function of the single variable \( t \). Find the value
of \( t \) that minimizes \( S \). Use this value of \( t \) to estimate \((x_2, y_2)\).

(e) Complete two more iterations of the process in part (d) to
obtain \((x_3, y_3)\). For this location of the distribution center
what is the sum of the distances to the retail outlets?

(f) Explain why \(-\nabla S(x, y)\) was used to approximate the
minimum value of \( S \). In what types of problems would you
use \(-\nabla S(x, y)\)?
In Exercises 27–30, (a) find the least squares regression line and (b) calculate $S$, the sum of the squared errors. Use the regression capabilities of a graphing utility to verify your results.

27. (0, 0), (1, 1), (3, 4), (4, 2), (5, 5)
28. (0, 4), (1, 3), (1, 1), (2, 0)
29. (0, 4), (1, 3), (2, 0)
30. (0, 4), (1, 3), (2, 0), (3, 1), (4, 1), (5, 2), (6, 2)

In Exercises 31–34, find the least squares regression line for the points. Use the regression capabilities of a graphing utility to verify your results. Use the graphing utility to plot the points and graph the regression line.

31. (0, 0), (1, 1), (3, 4), (4, 2), (5, 5)
32. (1, 0), (3, 3), (5, 6)
33. (0, 6), (4, 3), (5, 0), (8, −4), (10, −5)
34. (6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8)

35. **Modeling Data** The ages $x$ (in years) and systolic blood pressures $y$ of seven men are shown in the table.

<table>
<thead>
<tr>
<th>Age, $x$</th>
<th>16</th>
<th>25</th>
<th>39</th>
<th>45</th>
<th>49</th>
<th>64</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Systolic Blood Pressure, $y$</td>
<td>109</td>
<td>122</td>
<td>143</td>
<td>132</td>
<td>199</td>
<td>185</td>
<td>199</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.
(b) Use a graphing utility to plot the data and graph the model.
(c) Use the model to approximate the change in systolic blood pressure for each one-year increase in age.

36. **Modeling Data** A store manager wants to know the demand $y$ for an energy bar as a function of price $x$. The daily sales for three different prices of the energy bar are shown in the table.

<table>
<thead>
<tr>
<th>Price, $x$</th>
<th>$1.00$</th>
<th>$1.25$</th>
<th>$1.50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand, $y$</td>
<td>450</td>
<td>375</td>
<td>330</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.
(b) Use the model to estimate the demand when the price is $1.40$.

37. **Modeling Data** An agronomist used four test plots to determine the relationship between the wheat yield $y$ (in bushels per acre) and the amount of fertilizer $x$ (in hundreds of pounds per acre). The results are shown in the table.

<table>
<thead>
<tr>
<th>Fertilizer, $x$</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield, $y$</td>
<td>32</td>
<td>41</td>
<td>48</td>
<td>53</td>
</tr>
</tbody>
</table>

Use the regression capabilities of a graphing utility to find the least squares regression line for the data, and estimate the yield for a fertilizer application of 160 pounds per acre.

38. **Modeling Data** The table shows the percents $x$ and numbers $y$ (in millions) of women in the labor force for selected years (Source: U.S. Bureau of Labor Statistics).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent, $x$</td>
<td>39.3</td>
<td>43.3</td>
<td>46.3</td>
<td>51.5</td>
</tr>
<tr>
<td>Number, $y$</td>
<td>26.2</td>
<td>31.5</td>
<td>37.5</td>
<td>45.5</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.
(b) According to this model, approximately how many women enter the labor force for each one-point increase in the percent of women in the labor force?

39. Find a system of equations whose solution yields the coefficients $a$, $b$, and $c$ for the least squares regression quadratic $y = ax^2 + bx + c$ for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by minimizing the sum

$$S(a, b, c) = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2.$$ 

40. Use the Second Partials Test to verify that the formulas for $a$ and $b$ given in Theorem 13.18 yield a minimum.

**Hint:** Use the fact that $n \sum_{i=1}^{n} x_i^2 \geq \left( \sum_{i=1}^{n} x_i \right)^2$.
In Exercises 41–44, use the result of Exercise 39 to find the least squares regression quadratic for the given points. Use the regression capabilities of a graphing utility to confirm your results. Use the graphing utility to plot the points and graph the least squares regression quadratic.

41. \((-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)\)

42. \((-4, 5), (-2, 6), (2, 6), (4, 2)\)

43. \((0, 0), (2, 2), (3, 6), (4, 12)\)

44. \((0, 10), (1, 9), (2, 6), (3, 0)\)

45. **Modeling Data** After a new turbocharger for an automobile engine was developed, the following experimental data were obtained for speed \(y\) in miles per hour at two-second time intervals \(x\).

<table>
<thead>
<tr>
<th>Time, (x)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed, (y)</td>
<td>0</td>
<td>15</td>
<td>30</td>
<td>50</td>
<td>65</td>
<td>70</td>
</tr>
</tbody>
</table>

(a) Find a least squares regression quadratic for the data. Use a graphing utility to confirm your results.

(b) Use a graphing utility to plot the points and graph the model.

46. **Modeling Data** The table shows the world populations \(y\) (in billions) for five different years. (Source: U.S. Bureau of the Census, International Data Base)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population, (y)</td>
<td>5.6</td>
<td>5.8</td>
<td>5.9</td>
<td>6.1</td>
<td>6.2</td>
</tr>
</tbody>
</table>

Let \(x = 4\) represent the year 1994.

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the regression capabilities of a graphing utility to find the least squares regression quadratic for the data.

(c) Use a graphing utility to plot the data and graph the models.

(d) Use both models to forecast the world population for the year 2010. How do the two models differ as you extrapolate into the future?

47. **Modeling Data** A meteorologist measures the atmospheric pressure \(P\) (in kilograms per square meter) at altitude \(h\) (in kilometers). The data are shown below.

<table>
<thead>
<tr>
<th>Altitude, (h)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure, (P)</td>
<td>10,332</td>
<td>5583</td>
<td>2376</td>
<td>1240</td>
<td>517</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find a least squares regression line for the points \((h, \ln P)\).

(b) The result in part (a) is an equation of the form \(\ln P = ah + b\). Write this logarithmic form in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) If your graphing utility can fit logarithmic models to data, use it to verify the result in part (b).

48. **Modeling Data** The endpoints of the interval over which distinct vision is possible are called the near point and far point of the eye. With increasing age, these points normally change. The table shows the approximate near points in inches for various ages \(x\) (in years).

<table>
<thead>
<tr>
<th>Age, (x)</th>
<th>16</th>
<th>32</th>
<th>44</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Near Point, (y)</td>
<td>3.0</td>
<td>4.7</td>
<td>9.8</td>
<td>19.7</td>
<td>39.4</td>
</tr>
</tbody>
</table>

(a) Find a rational model for the data by taking the reciprocal of the near points to generate the points \((x, 1/y)\). Use the regression capabilities of a graphing utility to find a least squares regression line for the revised data. The resulting line has the form

\[
\frac{1}{y} = ax + b.
\]

Solve for \(y\).

(b) Use a graphing utility to plot the data and graph the model.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.


**Lagrange Multipliers**

Many optimization problems have restrictions, or *constraints*, on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain. In this section, you will study an ingenious technique for solving such problems. It is called the **Method of Lagrange Multipliers**.

To see how this technique works, suppose you want to find the rectangle of maximum area that can be inscribed in the ellipse given by

\[ \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1. \]

Let \((x, y)\) be the vertex of the rectangle in the first quadrant, as shown in Figure 13.77. Because the rectangle has sides of lengths \(2x\) and \(2y\), its area is given by

\[ f(x, y) = 4xy. \quad \text{Objective function} \]

You want to find \(x\) and \(y\) such that \(f(x, y)\) is a maximum. Your choice of \((x, y)\) is restricted to first-quadrant points that lie on the ellipse

\[ \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1. \quad \text{Constraint} \]

Now, consider the constraint equation to be a fixed level curve of

\[ g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}. \]

The level curves of \(f\) represent a family of hyperbolas

\[ f(x, y) = 4xy = k. \]

In this family, the level curves that meet the given constraint correspond to the hyperbolas that intersect the ellipse. Moreover, to maximize \(f(x, y)\), you want to find the hyperbola that just barely satisfies the constraint. The level curve that does this is the one that is *tangent* to the ellipse, as shown in Figure 13.78.
To find the appropriate hyperbola, use the fact that two curves are tangent at a point if and only if their gradient vectors are parallel. This means that \( \nabla f(x, y) \) must be a scalar multiple of \( \nabla g(x, y) \) at the point of tangency. In the context of constrained optimization problems, this scalar is denoted by \( \lambda \) (the lowercase Greek letter lambda).

\[
\nabla f(x, y) = \lambda \nabla g(x, y)
\]

The scalar \( \lambda \) is called a Lagrange multiplier. Theorem 13.19 gives the necessary conditions for the existence of such multipliers.

**THEOREM 13.19** Lagrange’s Theorem

Let \( f \) and \( g \) have continuous first partial derivatives such that \( f \) has an extremum at a point \((x_0, y_0)\) on the smooth constraint curve \( g(x, y) = c \). If \( \nabla g(x_0, y_0) \neq 0 \), then there is a real number \( \lambda \) such that

\[
\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).
\]

**Proof** To begin, represent the smooth curve given by \( g(x, y) = c \) by the vector-valued function

\[
\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad \mathbf{r}'(t) \neq 0
\]

where \( x' \) and \( y' \) are continuous on an open interval \( I \). Define the function \( h \) as

\[
h(t) = f(x(t), y(t)).
\]

Then, because \( f(x_0, y_0) \) is an extreme value of \( f \), you know that

\[
h(t_0) = f(x(t_0), y(t_0)) = f(x_0, y_0)
\]

is an extreme value of \( h \). This implies that \( h'(t_0) = 0 \), and, by the Chain Rule,

\[
h'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) = \nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0) = 0.
\]

So, \( \nabla f(x_0, y_0) \) is orthogonal to \( \mathbf{r}'(t_0) \). Moreover, by Theorem 13.12, \( \nabla g(x_0, y_0) \) is also orthogonal to \( \mathbf{r}'(t_0) \). Consequently, the gradients \( \nabla f(x_0, y_0) \) and \( \nabla g(x_0, y_0) \) are parallel, and there must exist a scalar \( \lambda \) such that

\[
\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).
\]

The Method of Lagrange Multipliers uses Theorem 13.19 to find the extreme values of a function \( f \) subject to a constraint.

**Method of Lagrange Multipliers**

Let \( f \) and \( g \) satisfy the hypothesis of Lagrange’s Theorem, and let \( f \) have a minimum or maximum subject to the constraint \( g(x, y) = c \). To find the minimum or maximum of \( f \), use the following steps.

1. Simultaneously solve the equations \( \nabla f(x, y) = \lambda \nabla g(x, y) \) and \( g(x, y) = c \) by solving the following system of equations.

\[
\begin{align*}
f_x(x, y) &= \lambda g_x(x, y) \\
f_y(x, y) &= \lambda g_y(x, y) \\
g(x, y) &= c
\end{align*}
\]

2. Evaluate \( f \) at each solution point obtained in the first step. The largest value yields the maximum of \( f \) subject to the constraint \( g(x, y) = c \), and the smallest value yields the minimum of \( f \) subject to the constraint \( g(x, y) = c \).
Constrained Optimization Problems

In the problem at the beginning of this section, you wanted to maximize the area of a rectangle that is inscribed in an ellipse. Example 1 shows how to use Lagrange multipliers to solve this problem.

EXAMPLE 1  Using a Lagrange Multiplier with One Constraint

Find the maximum value of \( f(x, y) = 4xy \) where \( x > 0 \) and \( y > 0 \), subject to the constraint \( (x^2/3^2) + (y^2/4^2) = 1 \).

Solution  To begin, let

\[
g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.
\]

By equating \( \nabla f(x, y) = 4y\mathbf{i} + 4x\mathbf{j} \) and \( \lambda \nabla g(x, y) = (2\lambda x/9)\mathbf{i} + (\lambda y/8)\mathbf{j} \), you can obtain the following system of equations.

\[
\begin{align*}
4y &= \frac{2}{9} \lambda x \\
4x &= \frac{1}{8} \lambda y \\
\frac{x^2}{3^2} + \frac{y^2}{4^2} &= 1 \quad \text{Constraint}
\end{align*}
\]

From the first equation, you obtain \( \lambda = 18y/x \), and substitution into the second equation produces

\[
4x = \frac{1}{8} \left( \frac{18y}{x} \right) y \quad \Longrightarrow \quad x^2 = \frac{9}{16} y^2.
\]

Substituting this value for \( x^2 \) into the third equation produces

\[
\frac{1}{9} \left( \frac{9}{16} y^2 \right) + \frac{1}{16} y^2 = 1 \quad \Longrightarrow \quad y^2 = 8.
\]

So, \( y = \pm 2\sqrt{2} \). Because it is required that \( y > 0 \), choose the positive value and find that

\[
\begin{align*}
x^2 &= \frac{9}{16} y^2 \\
&= \frac{9}{16} (8) = \frac{9}{2} \\
x &= \frac{3}{\sqrt{2}}
\end{align*}
\]

So, the maximum value of \( f \) is

\[
f \left( \frac{3}{\sqrt{2}}, 2\sqrt{2} \right) = 4xy = 4 \left( \frac{3}{\sqrt{2}} \right) \left( 2\sqrt{2} \right) = 24.
\]

NOTE  Example 1 can also be solved using the techniques you learned in Chapter 3. To see how, try to find the maximum value of \( A = 4xy \) given that

\[
\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.
\]

To begin, solve the second equation for \( y \) to obtain

\[
y = \frac{4}{3} \sqrt{9 - x^2}.
\]

Then substitute into the first equation to obtain

\[
A = 4x \left( \frac{4}{3} \sqrt{9 - x^2} \right).
\]

Finally, use the techniques of Chapter 3 to maximize \( A \).

Try It  Exploration A  Exploration B

Note that writing the constraint as

\[
g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \quad \text{or} \quad g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} - 1 = 0
\]

does not affect the solution—the constant is eliminated when you form \( \nabla g \).
EXAMPLE 2 A Business Application

The Cobb-Douglas production function (see Example 5, Section 13.1) for a software manufacturer is given by

\[ f(x, y) = 100x^{3/4}y^{1/4} \]  

Objective function

where \( x \) represents the units of labor (at $150 per unit) and \( y \) represents the units of capital (at $250 per unit). The total cost of labor and capital is limited to $50,000. Find the maximum production level for this manufacturer.

Solution From the given function, you have

\[ \nabla f(x, y) = 75x^{-1/4}y^{1/4} \mathbf{i} + 25x^{3/4}y^{-3/4} \mathbf{j}. \]

The limit on the cost of labor and capital produces the constraint

\[ g(x, y) = 150x + 250y = 50,000. \]

So, \( \lambda \nabla g(x, y) = 150 \mathbf{i} + 250 \mathbf{j}. \) This gives rise to the following system of equations.

\[
75x^{-1/4}y^{1/4} = 150 \lambda \\
25x^{3/4}y^{-3/4} = 250 \lambda \\
150x + 250y = 50,000
\]

Constraint

By solving for \( \lambda \) in the first equation

\[
\lambda = \frac{75x^{-1/4}y^{1/4}}{150} = \frac{x^{-1/4}y^{1/4}}{2}
\]

and substituting into the second equation, you obtain

\[
25x^{3/4}y^{-3/4} = 250 \left( \frac{x^{-1/4}y^{1/4}}{2} \right)
\]

\[
25x = 125y. \quad \text{Multiply by } x^{1/4}y^{3/4}. \]

So, \( x = 5y. \) By substituting into the third equation, you have

\[
150(5y) + 250y = 50,000 \\
1000y = 50,000 \\
y = 50 \text{ units of capital} \\
x = 250 \text{ units of labor.}
\]

So, the maximum production level is

\[ f(250, 50) = 100(250)^{3/4}(50)^{1/4} \]

\[ \approx 16,719 \text{ product units.} \]

FOR FURTHER INFORMATION For more information on the use of Lagrange multipliers in economics, see the article “Lagrange Multiplier Problems in Economics” by John V. Baxley and John C. Moorhouse in *The American Mathematical Monthly*.

Economists call the Lagrange multiplier obtained in a production function the **marginal productivity of money**. For instance, in Example 2 the marginal productivity of money at \( x = 250 \) and \( y = 50 \) is

\[
\lambda = \frac{x^{-1/4}y^{1/4}}{2} = \frac{(250)^{-1/4}(50)^{1/4}}{2} \approx 0.334
\]

which means that for each additional dollar spent on production, an additional 0.334 unit of the product can be produced.
Example 3  Lagrange Multipliers and Three Variables

Find the minimum value of 
\[ f(x, y, z) = 2x^2 + y^2 + 3z^2 \quad \text{Objective function} \]
subject to the constraint \( 2x - 3y - 4z = 49 \).

**Solution**  Let \( g(x, y, z) = 2x - 3y - 4z = 49 \). Then, because 
\[
\nabla f(x, y, z) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}
\]
and 
\[
\lambda \nabla g(x, y, z) = 2\lambda \mathbf{i} + 3\lambda \mathbf{j} - 4\lambda \mathbf{k}
\]
you obtain the following system of equations.
\[
\begin{align*}
4x &= 2\lambda \\
2y &= -3\lambda \\
6z &= -4\lambda \\
2x - 3y - 4z &= 49
\end{align*}
\]
The solution of this system is \( x = 3, y = -9, \) and \( z = -4 \). So, the optimum value of \( f \) is
\[
f(3, -9, -4) = 2(3)^2 + (-9)^2 + 3(-4)^2
\]
\[
= 147.
\]
From the original function and constraint, it is clear that \( f(x, y, z) \) has no maximum. So, the optimum value of \( f \) determined above is a minimum.

Try It  Exploration A  Open Exploration

A graphical interpretation of constrained optimization problems in two variables was given at the beginning of this section. In three variables, the interpretation is similar, except that level surfaces are used instead of level curves. For instance, in Example 3, the level surfaces of \( f \) are ellipsoids centered at the origin, and the constraint 
\[
2x - 3y - 4z = 49
\]
is a plane. The minimum value of \( f \) is represented by the ellipsoid that is tangent to the constraint plane, as shown in Figure 13.79.

Example 4  Optimization Inside a Region

Find the extreme values of 
\[ f(x, y) = x^2 + 2y^2 - 2x + 3 \quad \text{Objective function} \]
subject to the constraint \( x^2 + y^2 \leq 10 \).

**Solution**  To solve this problem, you can break the constraint into two cases.

**a.** For points on the circle \( x^2 + y^2 = 10 \), you can use Lagrange multipliers to find that the maximum value of \( f(x, y) \) is 24—this value occurs at \((-1, 3)\) and at \((-1, -3)\). In a similar way, you can determine that the minimum value of \( f(x, y) \) is approximately 6.675—this value occurs at \((\sqrt{10}, 0)\).

**b.** For points inside the circle, you can use the techniques discussed in Section 13.8 to conclude that the function has a relative minimum of 2 at the point \((1, 0)\).

By combining these two results, you can conclude that \( f \) has a maximum of 24 at \(( -1, \pm 3 )\) and a minimum of 2 at \(( 1, 0 )\), as shown in Figure 13.80.

Try It  Exploration A
The Method of Lagrange Multipliers with Two Constraints

For optimization problems involving two constraint functions $g$ and $h$, you can introduce a second Lagrange multiplier, $\mu$ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where the gradient vectors are not parallel, as illustrated in Example 5.

**EXAMPLE 5**  Optimization with Two Constraints

Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 11$. Find the extreme temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and the sphere.

**Solution**  The two constraints are

$$g(x, y, z) = x^2 + y^2 + z^2 = 11 \quad \text{and} \quad h(x, y, z) = x + y + z = 3.$$ 

Using

$$\nabla T(x, y, z) = 2i + 2j + 2zk$$

$$\lambda \nabla g(x, y, z) = 2\lambda xi + 2\lambda yj + 2\lambda zk$$

and

$$\mu \nabla h(x, y, z) = \mu i + \mu j + \mu k$$

you can write the following system of equations.

$$2 = 2\lambda x + \mu \quad T_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z)$$

$$2 = 2\lambda y + \mu \quad T_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z)$$

$$2 = 2\lambda z + \mu \quad T_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z)$$

$$x^2 + y^2 + z^2 = 11 \quad \text{Constraint 1}$$

$$x + y + z = 3 \quad \text{Constraint 2}$$

By subtracting the second equation from the first, you can obtain the following system.

$$\lambda (x - y) = 0$$

$$2(1 - \lambda) - \mu = 0$$

$$x^2 + y^2 + z^2 = 11$$

$$x + y + z = 3$$

From the first equation, you can conclude that $\lambda = 0$ or $x = y$. If $\lambda = 0$, you can show that the critical points are $(3, -1, 1)$ and $(-1, 3, 1)$. (Try doing this—it takes a little work.) If $\lambda \neq 0$, then $x = y$ and you can show that the critical points occur when $x = y = (3 + 2\sqrt{3})/3$ and $z = (3 + 4\sqrt{3})/3$. Finally, to find the optimal solutions, compare the temperatures at the four critical points.

$$T(3, -1, 1) = T(-1, 3, 1) = 25$$

$$T\left(\frac{3 - 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}, \frac{3 + 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$

$$T\left(\frac{3 + 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}, \frac{3 - 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$

So, $T = 25$ is the minimum temperature and $T = \frac{91}{3}$ is the maximum temperature on the curve.

**Try It**  **Exploration A**
Exercises for Section 13.10

The symbol \[\textcolor{green}{\bullet}\] indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on [S] to view the complete solution of the exercise.

Click on [M] to print an enlarged copy of the graph.

In Exercises 1–4, identify the constraint and level curves of the objective function shown in the figure. Use the figure to approximate the indicated extrema, assuming that \(x\) and \(y\) are positive. Use Lagrange multipliers to verify your result.

1. Maximize \(z = xy\)
   Constraint: \(x + y = 10\)

2. Maximize \(z = xy\)
   Constraint: \(2x + y = 4\)

3. Minimize \(z = x^2 + y^2\)
   Constraint: \(x + y - 4 = 0\)

4. Minimize \(z = x^2 + y^2\)
   Constraint: \(2x + 4y = 5\)

In Exercises 5–12, use Lagrange multipliers to find the indicated extrema, assuming that \(x\) and \(y\) are positive.

5. Minimize \(f(x, y) = x^2 - y^2\)
   Constraint: \(x - 2y + 6 = 0\)

6. Maximize \(f(x, y) = x^2 - y^2\)
   Constraint: \(2y - x^2 = 0\)

7. Maximize \(f(x, y) = 2x + 2xy + y\)
   Constraint: \(2x + y = 100\)

8. Minimize \(f(x, y) = 3x + y + 10\)
   Constraint: \(x^2y = 6\)

9. Maximize \(f(x, y) = \sqrt{6 - x^2 - y^2}\)
   Constraint: \(x + y - 2 = 0\)

10. Minimize \(f(x, y) = \sqrt{x^2 + y^2}\)
    Constraint: \(2x + 4y - 15 = 0\)

11. Maximize \(f(x, y) = e^{xy}\)
    Constraint: \(x^2 + y^2 = 8\)

12. Minimize \(f(x, y) = 2x + y\)
    Constraint: \(xy = 32\)

In Exercises 13 and 14, use Lagrange multipliers to find any extrema of the function subject to the constraint \(x^2 + y^2 \leq 1\).

13. \(f(x, y) = x^2 + 3xy + y^2\)

14. \(f(x, y) = e^{-xy/4}\)

In Exercises 15–18, use Lagrange multipliers to find the indicated extrema, assuming that \(x\), \(y\), and \(z\) are positive.

15. Minimize \(f(x, y, z) = x^2 + y^2 + z^2\)
    Constraint: \(x + y + z = 0\)

16. Maximize \(f(x, y, z) = xyz\)
    Constraint: \(x + y + z = -6\)

17. Minimize \(f(x, y, z) = x^2 + y^2 + z^2\)
    Constraint: \(x + y + z = 1\)

18. Minimize \(f(x, y) = x^2 - 10x + y^2 - 14y + 70\)
    Constraint: \(x + y = 10\)

In Exercises 19–22, use Lagrange multipliers to find the indicated extrema of \(f\) subject to two constraints. In each case assume that \(x\), \(y\), and \(z\) are nonnegative.

19. Maximize \(f(x, y, z) = x^2 + 2xy + y^2\)
    Constraints: \(x + y + z = 32, \ x - y + z = 0\)

20. Minimize \(f(x, y, z) = x^2 + y^2 + z^2\)
    Constraints: \(x + 2z = 6, \ x + y = 12\)

21. Maximize \(f(x, y, z) = xy + yz\)
    Constraints: \(x + 2y = 6, \ x - 3z = 0\)

22. Maximize \(f(x, y, z) = x^2 + 2y^2 + z^2\)
    Constraints: \(x^2 + z^2 = 6, \ y - 2y = 0\)

In Exercises 23–26, use Lagrange multipliers to find the minimum distance from the curve or surface to the indicated point. [Hint: In Exercise 23, minimize \(f(x, y) = x^2 + y^2\) subject to the constraint \(2x + 3y = -1\].

<table>
<thead>
<tr>
<th>Curve</th>
<th>Point</th>
<th>Surface</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line: (2x + 3y = -1)</td>
<td>((0, 0))</td>
<td>Circle: ((x - 4)^2 + y^2 = 4)</td>
<td>((0, 10))</td>
</tr>
<tr>
<td>Plane: (x + y + z = 1)</td>
<td>((2, 1, 1))</td>
<td>Plane: (x + y = 2)</td>
<td>((4, 0, 0))</td>
</tr>
</tbody>
</table>

In Exercises 27 and 28, find the highest point on the curve of intersection of the surfaces.

27. Sphere: \(x^2 + y^2 + z^2 = 36\), Plane: \(2x + y - z = 2\)

28. Cone: \(x^2 + y^2 - z^2 = 0\), Plane: \(x + 2z = 4\)

Writing About Concepts

29. Explain what is meant by constrained optimization problems.

30. Explain the method of Lagrange multipliers for solving constrained optimization problems.
31. **Maximum Volume** Use Lagrange multipliers to find the dimensions of the rectangular package of largest volume subject to the constraint that the sum of the length and the girth cannot exceed 108 inches. Compare the answer with that obtained in Exercise 9, Section 13.9.

32. **Maximum Volume** The material for the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. Use Lagrange multipliers to find the dimensions of the box of largest volume that can be made for a fixed cost $C$. (Maximize $V = xyz$ subject to $1.5xy + 2xz + 2yz = C$.) Compare the answer to that obtained in Exercise 10, Section 13.9.

33. **Minimum Cost** A cargo container (in the shape of a rectangular solid) must have a volume of 480 cubic feet. The bottom will cost $5 per square foot to construct and the sides and the top will cost $3 per square foot to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has minimum cost.

34. **Minimum Surface Area** Use Lagrange multipliers to find the dimensions of a right circular cylinder with volume $V_0$ cubic units and minimum surface area.

35. **Maximum Volume** Use Lagrange multipliers to find the dimensions of a rectangular box of maximum volume that can be inscribed (with edges parallel to the coordinate axes) in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

36. **Geometric and Arithmetic Means**

(a) Use Lagrange multipliers to prove that the product of three positive numbers $x, y,$ and $z$, whose sum has the constant value $S$, is a maximum when the three numbers are equal. Use this result to prove that

$$\sqrt[3]{xyz} \leq \frac{x + y + z}{3}.$$

(b) Generalize the result of part (a) to prove that the product $x_1x_2x_3 \cdots x_n$ is a maximum when $x_1 = x_2 = x_3 = \cdots = x_n$, $n \sum_{j=1}^n x_j = S$, and all $x_j \geq 0$. Then prove that

$$\sqrt[n]{x_1x_2x_3 \cdots x_n} \leq \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

This shows that the geometric mean is never greater than the arithmetic mean.

37. **Refraction of Light** When light waves traveling in a transparent medium strike the surface of a second transparent medium, they tend to “bend” in order to follow the path of minimum time. This tendency is called refraction and is described by Snell’s Law of Refraction,

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2},$$

where $\theta_1$ and $\theta_2$ are the magnitudes of the angles shown in the figure, and $v_1$ and $v_2$ are the velocities of light in the two media. Use Lagrange multipliers to derive this law using $x + y = a$.

38. **Area and Perimeter** A semicircle is on top of a rectangle (see figure). If the area is fixed and the perimeter is a minimum, or if the perimeter is fixed and the area is a maximum, use Lagrange multipliers to verify that the length of the rectangle is twice its height.

39. **Hardy-Weinberg Law** Use Lagrange multipliers to maximize $P(p, q, r) = 2pq + 2pr + 2qr$ subject to $p + q + r = 1$. (See Exercise 20 in Section 13.9.)

40. **Temperature Distribution** Let $T(x, y, z) = 100 + x^2 + y^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 50$. Find the maximum temperature on the curve formed by the intersection of the sphere and the plane $x - z = 0$.

**Production Level** In Exercises 41 and 42, find the maximum production level $P$ if the total cost of labor ($at$ $48 per unit) and capital ($at$ $36 per unit) is limited to $100,000, where $x$ is the number of units of labor and $y$ is the number of units of capital.

41. $P(x, y) = 100x^{0.25}y^{0.75}$

42. $P(x, y) = 100x^{0.4}y^{0.6}$

**Cost** In Exercises 43 and 44, find the minimum cost of producing 20,000 units of a product, where $x$ is the number of units of labor ($at$ $48 per unit) and $y$ is the number of units of capital ($at$ $36 per unit).

43. $P(x, y) = 100x^{0.25}y^{0.75}$

44. $P(x, y) = 100x^{0.6}y^{0.4}$

45. **Investigation** Consider the objective function $g(x, \alpha, \beta, \gamma) = \cos \alpha \cos \beta \cos \gamma$ subject to the constraint that $\alpha, \beta,$ and $\gamma$ are the angles of a triangle.

(a) Use Lagrange multipliers to maximize $g$.

(b) Use the constraint to reduce the function $g$ to a function of two independent variables. Use a computer algebra system to graph the surface represented by $g$. Identify the maximum values on the graph.

**Putnam Exam Challenge**

46. A can buoy is to be made of three pieces, namely, a cylinder and two equal cones, the altitude of each cone being equal to the altitude of the cylinder. For a given area of surface, what shape will have the greatest volume?

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.
Review Exercises for Chapter 13

The symbol \( \infty \) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on [S] to view the complete solution of the exercise.
Click on [M] to print an enlarged copy of the graph.

In Exercises 1 and 2, use the graph to determine whether \( z \) is a function of \( x \) and \( y \). Explain.

1. [Rotatable Graph]

2. [Rotatable Graph]

In Exercises 3–6, use a computer algebra system to graph several level curves of the function.

3. \( f(x, y) = e^{x+y^2} \)
4. \( f(x, y) = \ln xy \)
5. \( f(x, y) = x^2 - y^2 \)
6. \( f(x, y) = \frac{x}{x + y} \)

In Exercises 7 and 8, use a computer algebra system to graph the function.

7. \( f(x, y) = e^{-(x+y)^2} \)
8. \( g(x, y) = |y|^{1+|y|} \)

In Exercises 9 and 10, sketch the graph of the level surface \( f(x, y, z) = c \) at the given value of \( c \).

9. \( f(x, y, z) = x^2 - y + z^2 \), \( c = 1 \)
10. \( f(x, y, z) = 9x^2 - y^3 + 9z^2 \), \( c = 0 \)

In Exercises 11–14, find the limit and discuss the continuity of the function (if it exists).

11. \( \lim_{(x, y) \to (1, 1)} \frac{xy}{x^2 + y^2} \)
12. \( \lim_{(x, y) \to (1, 1)} \frac{xy}{x^2 - y^2} \)
13. \( \lim_{(x, y) \to (0, 0)} \frac{-4x^2y}{x^4 + y^2} \)
14. \( \lim_{(x, y) \to (0, 0)} \frac{y + xe^{-y^2}}{1 + x^2} \)

In Exercises 15–24, find all first partial derivatives.

15. \( f(x, y) = e^x \cos y \)
16. \( f(x, y) = \frac{xy}{x + y} \)
17. \( f(x, y) = 4x^2 + 2y^3 - 3z \)
18. \( f(x, y) = 2x^2 + 3x^2 + 4y^3 \)
19. \( g(x, y) = \frac{xy}{x^2 + y^2} \)
20. \( w = \sqrt{x^2 + y^2 + z^2} \)
21. \( f(x, y, z) = \arctan \frac{y}{x} \)
22. \( f(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}} \)
23. \( u(x, t) = e^{-n^2t} \sin nx \)
24. \( u(x, t) = c \sin(akx) \cos kt \)

25. Think About It Sketch a graph of a function \( z = f(x, y) \) whose derivative \( f_x \) is always negative and whose derivative \( f_y \) is always negative.

26. Find the slopes of the surface \( z = x^2 \ln(y + 1) \) in the \( x \)- and \( y \)-directions at the point \((2, 0, 0)\).

In Exercises 27–30, find all second partial derivatives and verify that the second mixed partials are equal.

27. \( f(x, y) = 3x^2 - xy + 2y^3 \)
28. \( h(x, y) = \frac{x}{x + y} \)
29. \( h(x, y) = x \sin y + y \cos x \)
30. \( g(x, y) = \cos(x - 2y) \)

Laplace’s Equation In Exercises 31–34, show that the function satisfies Laplace’s equation

\[ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0. \]

31. \( z = x^2 - y^2 \)
32. \( z = x^2 - 3xy^2 \)
33. \( z = \frac{y}{x^2 + y^2} \)
34. \( z = e^{x} \sin y \)

In Exercises 35 and 36, find the total differential.

35. \( z = x \sin \frac{y}{x} \)
36. \( z = \frac{xy}{\sqrt{x^2 + y^2}} \)

37. Error Analysis The legs of a right triangle are measured to be 5 centimeters and 12 centimeters, with a possible error of \( \frac{1}{2} \) centimeter. Approximate the maximum possible error in computing the length of the hypotenuse. Approximate the maximum percent error.

38. Error Analysis To determine the height of a tower, the angle of elevation to the top of the tower was measured from a point 100 feet \( \frac{1}{2} \) foot from the base. The angle is measured at \( 33^\circ \) with a possible error of \( 1^\circ \). Assuming that the ground is horizontal, approximate the maximum error in determining the height of the tower.

39. Volume A right circular cone is measured and the radius and height are found to be 2 inches and 5 inches, respectively. The possible error in measurement is \( \frac{1}{2} \) inch. Approximate the maximum possible error in the computation of the volume.

40. Lateral Surface Area Approximate the error in the computation of the lateral surface area of the cone in Exercise 39. (The lateral surface area is given by \( A = \pi \sqrt{r^2 + h^2} \).)
In Exercises 41–44, find the indicated derivatives (a) using
the appropriate Chain Rule and (b) by substitution before
differentiating.

41. \( w = \ln(x^2 + y^3), \frac{dw}{dt} \)

\( x = 2t + 3, \quad y = 4 - t \)

42. \( u = y^2 - x, \frac{du}{dt} \)

\( x = \cos t, \quad y = \sin t \)

43. \( u = x^2 + y^2 + z^2, \frac{\partial u}{\partial r} \frac{du}{dt} \)

\( x = r \cos t, \quad y = r \sin t, \quad z = t \)

44. \( w = \frac{xy}{z}, \frac{\partial w}{\partial r}, \frac{\partial w}{\partial t} \)

\( x = 2r + t, \quad y = rt, \quad z = 2r - t \)

In Exercises 45 and 46, differentiate implicitly to find the first
partial derivatives of \( z \).

45. \( x^2 y - 2y z - x z - z^2 = 0 \)

46. \( x z^2 - y \sin z = 0 \)

In Exercises 47–50, find the directional derivative of the function
at \( P \) in the direction of \( v \).

47. \( f(x, y) = x^2 y, \quad v = i - j \)

48. \( f(x, y) = \frac{1}{2} y^2 - x^2, \quad v = 2i + j \)

49. \( w = y^2 + x z, \quad (1, 2, 2), \quad v = 2i - j + 2k \)

50. \( w = 6x^2 + 3xy - 4y^2 z, \quad (1, 0, 1), \quad v = i + j - k \)

In Exercises 51–54, find the gradient of the function and the
maximum value of the directional derivative at the given point.

51. \( z = \frac{y}{x^2 + y^2}, \quad (1, 1) \)

52. \( z = \frac{x^2}{y}, \quad (2, 1) \)

53. \( z = e^{-x} \cos y, \quad (0, \pi/4) \)

54. \( z = x^2 y, \quad (2, 1) \)

In Exercises 55 and 56, use the gradient to find a unit normal
to the graph of the equation at the given point.

55. \( 9x^2 - 4y^2 = 65, \quad (3, 2) \)

56. \( 4y \sin x - y^2 = 3, \quad (\pi/2, 1) \)

In Exercises 57–60, find an equation of the tangent plane and
parametric equations of the normal line to the surface at the
given point.

<table>
<thead>
<tr>
<th>Surface</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>57. ( f(x, y) = x^2 y )</td>
<td>(2, 1, 4)</td>
</tr>
<tr>
<td>58. ( f(x, y) = \sqrt{25 - y^2} )</td>
<td>(2, 3, 4)</td>
</tr>
<tr>
<td>59. ( z = -9 + 4x - 6y - x^2 - y^2 )</td>
<td>(2, -3, 4)</td>
</tr>
<tr>
<td>60. ( z = \sqrt{9 - x^2 - y^2} )</td>
<td>(1, 2, 2)</td>
</tr>
</tbody>
</table>

In Exercises 61 and 62, find symmetric equations of the tangent
plane to the curve of intersection of the surfaces at the given point.

<table>
<thead>
<tr>
<th>Surfaces</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>61. ( z = x^2 - y^2 ), ( z = 3 )</td>
<td>(2, 1, 3)</td>
</tr>
<tr>
<td>62. ( z = 25 - y^2 ), ( y = x )</td>
<td>(4, 4, 9)</td>
</tr>
</tbody>
</table>

63. Find the angle of inclination \( \theta \) of the tangent plane to the
surface \( x^2 + y^2 + z^2 = 14 \) at the point (2, 1, 3).

64. Approximation

Consider the following approximations for \( f(x, y) \) function centered at (0, 0).

**Linear approximation:**

\[ P_1(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y \]

**Quadratic approximation:**

\[ P_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2 \]

(a) Find the linear approximation of \( f(x, y) = \cos x + \sin y \)
centered at (0, 0).

(b) Find the quadratic approximation of \( f(x, y) = \cos x + \sin y \)
centered at (0, 0).

(c) If \( y = 0 \) in the quadratic approximation, you obtain the
second-degree Taylor polynomial for what function?

(d) Complete the table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( f(x, y) )</th>
<th>( P_1(x, y) )</th>
<th>( P_2(x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>0.3</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

(e) Use a computer algebra system to graph the surface:
\( z = f(x, y), \quad z = P_1(x, y), \quad z = P_2(x, y) \). How does the
accuracy of the approximations change as the distance from
(0, 0) increases?

In Exercises 65–68, examine the function for relative extrema.

Use a computer algebra system to graph the function and con-
firm your results.

65. \( f(x, y) = x^3 - 3xy + y^2 \)

66. \( f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14 \)

67. \( f(x, y) = xy + \frac{1}{x} + \frac{1}{y} \)

68. \( z = 50(x + y) + (0.1x^3 + 20x + 150) = (0.05y^3 + 20.6y + 125) \)
71. **Maximum Profit** A corporation manufactures digital cameras at two locations. The cost functions for producing \( x_1 \) units at location 1 and \( x_2 \) units at location 2 are
\[
C_1 = 0.05x_1^2 + 15x_1 + 5400 \\
C_2 = 0.03x_2^2 + 15x_2 + 6100
\]
and the total revenue function is
\[
R = [225 - 0.4(x_1 + x_2)](x_1 + x_2).
\]
Find the production levels at the two locations that will maximize the profit \( P(x_1, x_2) = R - C_1 - C_2 \).

72. **Minimum Cost** A manufacturer has an order for 1000 units of wooden benches that can be produced at two locations. Let \( x_1 \) and \( x_2 \) be the numbers of units produced at the two locations. The cost function is
\[
C = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2.
\]
Find the number that should be produced at each location to meet the order and minimize cost.

73. **Production Level** The production function for a candy manufacturer is
\[
f(x, y) = 4x + xy + 2y
\]
where \( x \) is the number of units of labor and \( y \) is the number of units of capital. Assume that the total amount available for labor and capital is $2000, and that units of labor and capital cost $20 and $4, respectively. Find the maximum production level for this manufacturer.

74. Find the minimum distance from the point \((2, 2, 0)\) to the surface \( z = x^2 + y^2 \).

75. **Modeling Data** The data in the table show the yield \( y \) (in milligrams) of a chemical reaction after \( t \) minutes.

<table>
<thead>
<tr>
<th>Minutes, ( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield, ( y )</td>
<td>1.5</td>
<td>7.4</td>
<td>10.2</td>
<td>13.4</td>
</tr>
<tr>
<td>Minutes, ( t )</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>Yield, ( y )</td>
<td>15.8</td>
<td>16.3</td>
<td>18.2</td>
<td>18.3</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data. Then use the graphing utility to plot the data and graph the model.

(b) Use a graphing utility to plot the points \((\ln t, y)\). Do these points appear to follow a linear pattern more closely than the plot of the given data in part (a)?

(c) Use the regression capabilities of a graphing utility to find the least squares regression line for the points \((\ln t, y)\) and obtain the logarithmic model \( y = a + b \ln t \).

(d) Use a graphing utility to plot the data and graph the linear and logarithmic models. Which is a better model? Explain.

76. **Modeling Data** The table shows the drag force \( y \) in kilograms for a motor vehicle at indicated speeds \( x \) in kilometers per hour.

<table>
<thead>
<tr>
<th>Speed, ( x )</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>100</th>
<th>125</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drag, ( y )</td>
<td>28</td>
<td>38</td>
<td>54</td>
<td>75</td>
<td>102</td>
</tr>
</tbody>
</table>

(a) Use the regression capabilities of a graphing utility to find the least squares regression quadratic for the data.

(b) Use the model to estimate the total drag when the vehicle is moving at 80 kilometers per hour.

In Exercises 77 and 78, use Lagrange multipliers to locate and classify any extrema of the function.

77. \( w = xy + yz + xz \)
Constraint: \( x + y + z = 1 \)

78. \( z = x^2y \)
Constraint: \( x + 2y = 2 \)

79. **Minimum Cost** A water line is to be built from point \( P \) to point \( S \) and must pass through regions where construction costs differ (see figure). The cost per kilometer in dollars is 3\( k \) from \( P \) to \( Q \), 2\( k \) from \( Q \) to \( R \), and \( k \) from \( R \) to \( S \). For simplicity, let \( k = 1 \). Use Lagrange multipliers to find \( x \), \( y \), and \( z \) such that the total cost \( C \) will be minimized.

80. **Investigation** Consider the objective function \( f(x, y) = ax + by \) subject to the constraint \( x^2/64 + y^2/36 = 1 \). Assume that \( x \) and \( y \) are positive.

(a) Use a computer algebra system to graph the constraint. If \( a = 4 \) and \( b = 3 \), use the computer algebra system to graph the level curves of the objective function. By trial and error find the level curve that appears to be tangent to the ellipse. Use the result to approximate the maximum of \( f \) subject to the constraint.

(b) Repeat part (a) for \( a = 4 \) and \( b = 9 \).
P.S. Problem Solving

The symbol ✱ indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on [S] to view the complete solution of the exercise.

Click on [M] to print an enlarged copy of the graph.

1. Heron’s Formula states that the area of a triangle with sides of lengths $a, b, c$ is given by
   \[ A = \sqrt{s(s-a)(s-b)(s-c)} \]
   where $s = \frac{a + b + c}{2}$, as shown in the figure.

(a) Use Heron’s Formula to find the area of the triangle with vertices $(0, 0), (3, 4),$ and $(6, 0)$.

(b) Show that among all triangles having a fixed perimeter, the triangle with the largest area is an equilateral triangle.

(c) Show that among all triangles having a fixed area, the triangle with the smallest perimeter is an equilateral triangle.

2. An industrial container is in the shape of a cylinder with hemispherical ends, as shown in the figure. The container must hold 1000 liters of fluid. Determine the radius $r$ and length $h$ that minimize the amount of material used in the construction of the tank.

3. Let $P(x_0, y_0, z_0)$ be a point in the first octant on the surface $xyz = 1$.
   (a) Find the equation of the tangent plane to the surface at the point $P$.
   (b) Show that the volume of the tetrahedron formed by the three coordinate planes and the tangent plane is constant, independent of the point of tangency (see figure).

4. Use a graphing utility to graph the functions $f(x) = \sqrt{x^3 - 1}$ and $g(x) = x$ in the same viewing window.
   (a) Show that
   \[ \lim_{x \to -\infty} [f(x) - g(x)] = 0 \quad \text{and} \quad \lim_{x \to -\infty} [f(x) - g(x)] = 0. \]
   (b) Find the point on the graph of $f$ that is farthest from the line $y = x$.

5. Consider the function
   \[ f(x, y) = \begin{cases} \frac{4xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \]
   and the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}(i+j)$.

Does the directional derivative of $f$ at $P(0, 0)$ in the direction of $\mathbf{u}$ exist? If $f(0, 0)$ were defined as 2 instead of 0, would the directional derivative exist?

6. A heated storage room is shaped like a rectangular box and has a volume of 1000 cubic feet, as shown in the figure. Because warm air rises, the heat loss per unit of area through the ceiling is five times as great as the heat loss through the floor. The heat loss through the four walls is three times as great as the heat loss through the floor. Determine the room dimensions that will minimize heat loss and therefore minimize heating costs.

7. Repeat Exercise 6 assuming that the heat loss through the walls and ceiling remain the same, but the floor is insulated so that there is no heat loss through the floor.

8. Consider a circular plate of radius 1 given by $x^2 + y^2 \leq 1$, as shown in the figure. The temperature at any point $P(x, y)$ on the plate is $T(x, y) = 2x^2 + y^2 - y + 10$.

(a) Sketch the isotherm $T(x, y) = 10$. To print an enlarged copy of the graph, select the MathGraph button.

(b) Find the hottest and coldest points on the plate.

9. Consider the Cobb-Douglas production function
   \[ f(x, y) = ax^{\beta}y^{1-\beta}, \quad 0 < a < 1 \]
   (a) Show that $f$ satisfies the equation $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f$.
   (b) Show that $f(tx, ty) = tf(x, y)$.

10. Rewrite Laplace’s equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ in cylindrical coordinates.
11. A projectile is launched at an angle of 45° with the horizontal and with an initial velocity of 64 feet per second. A television camera is located in the plane of the path of the projectile 50 feet behind the launch site (see figure).

(a) Find parametric equations for the path of the projectile in terms of the parameter t representing time.
(b) Write the angle α that the camera makes with the horizontal in terms of x and y and in terms of t.
(c) Use the results of part (b) to find \( \frac{d\alpha}{dt} \).
(d) Use a graphing utility to graph α in terms of t. Is the graph symmetric to the axis of the parabolic arch of the projectile? At what time is the rate of change of α greatest?
(e) At what time is the angle α maximum? Does this occur when the projectile is at its greatest height?

12. Consider the distance d between the launch site and the projectile in Exercise 11.
(a) Write the distance d in terms of x and y and in terms of the parameter t.
(b) Use the results of part (a) to find the rate of change of d.
(c) Find the rate of change of the distance when \( t = 2 \).
(d) When is the rate of change of d minimum during the flight of the projectile? Does this occur at the time when the projectile reaches its maximum height?

13. Consider the function
\[
f(x, y) = (\alpha x^2 + \beta y^2)e^{-(x^2+y^2)}, \quad 0 < |\alpha| < \beta.
\]
(a) Use a computer algebra system to graph the function for \( \alpha = 1 \) and \( \beta = 2 \), and identify any extrema or saddle points.
(b) Use a computer algebra system to graph the function for \( \alpha = -1 \) and \( \beta = 2 \), and identify any extrema or saddle points.
(c) Generalize the results in parts (a) and (b) for the function f.

14. Prove that if f is a differentiable function such that
\[
\nabla f(x_0, y_0) = 0
\]
then the tangent plane at \( (x_0, y_0) \) is horizontal.

15. The figure shows a rectangle that is approximately \( l = 6 \) centimeters long and \( h = 1 \) centimeter high.

(a) Draw a rectangular strip along the rectangular region showing a small increase in length.
(b) Draw a rectangular strip along the rectangular region showing a small increase in height.
(c) Use the results in parts (a) and (b) to identify the measurement that has more effect on the area \( A \) of the rectangle.
(d) Verify your answer in part (c) analytically by comparing the value of \( dA \) when \( dl = 0.01 \) and when \( dh = 0.01 \).

16. Consider converting a point \( (5 \pm 0.05, \pi/18 \pm 0.05) \) in polar coordinates to rectangular coordinates \((x, y)\). (This equation describes the small transverse vibration of an elastic string such as those on certain musical instruments.)

(a) Use a geometric argument to determine whether the accuracy in \( x \) is more dependent on the accuracy in \( r \) or on the accuracy in \( \theta \). Explain.
(b) Use a geometric argument to determine whether the accuracy in \( y \) is more dependent on the accuracy in \( r \) or on the accuracy in \( \theta \). Explain.
(c) Verify your answer in part (a) and (b) by identifying the measurement that has more effect on the area of the ellipse.

17. Let \( f \) be a differentiable function of one variable. Show that all tangent planes to the surface \( z = yf(x/y) \) intersect in a common point.

18. Consider the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
that encloses the circle \( x^2 + y^2 = 2a \). Find values of \( a \) and \( \ell \) that minimize the area of the ellipse.

19. Show that
\[
u(x, t) = \frac{1}{2} \left[ \sin(x - t) + \sin(x + t) \right]
\]
is a solution to the one-dimensional wave equation
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.
\]

20. Show that
\[
u(x, t) = \frac{1}{2} \left[ f(x - ct) + f(x + ct) \right]
\]
is a solution to the one-dimensional wave equation
\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.
\]

(This equation describes the small transverse vibration of an elastic string such as those on certain musical instruments.)