Section 15.1 Vector Fields

- Understand the concept of a vector field.
- Determine whether a vector field is conservative.
- Find the curl of a vector field.
- Find the divergence of a vector field.

**Vector Fields**

In Chapter 12, you studied vector-valued functions—functions that assign a vector to a real number. There you saw that vector-valued functions of real numbers are useful in representing curves and motion along a curve. In this chapter, you will study two other types of vector-valued functions—functions that assign a vector to a point in the plane or a point in space. Such functions are called vector fields, and they are useful in representing various types of force fields and velocity fields.

**Definition of Vector Field**

Let $M$ and $N$ be functions of two variables $x$ and $y$, defined on a plane region $R$. The function $\mathbf{F}$ defined by

$$
\mathbf{F}(x, y) = Mi + Nj
$$

is called a **vector field over $R$.**

Let $M$, $N$, and $P$ be functions of three variables $x$, $y$, and $z$, defined on a solid region $Q$ in space. The function $\mathbf{F}$ defined by

$$
\mathbf{F}(x, y, z) = Mi + Nj +Pk
$$

is called a **vector field over $Q$.**

From this definition you can see that the gradient is one example of a vector field. For example, if

$$
f(x, y) = x^2 + y^2
$$

then the gradient of $f$

$$
\nabla f(x, y) = f_x(x, y)i + f_y(x, y)j = 2xi + 2yj
$$

is a vector field in the plane. From Chapter 13, the graphical interpretation of this field is a family of vectors, each of which points in the direction of maximum increase along the surface given by $z = f(x, y)$. For this particular function, the surface is a paraboloid and the gradient tells you that the direction of maximum increase along the surface is the direction given by the ray from the origin through the point $(x, y)$.

Similarly, if

$$
f(x, y, z) = x^2 + y^2 + z^2
$$

then the gradient of $f$

$$
\nabla f(x, y, z) = f_x(x, y, z)i + f_y(x, y, z)j + f_z(x, y, z)k
\quad = 2xi + 2yj + 2zk
$$

is a vector field in space.

A vector field is **continuous** at a point if each of its component functions $M$, $N$, and $P$ is continuous at that point.
Some common physical examples of vector fields are velocity fields, gravitational fields, and electric force fields.

1. Velocity fields describe the motions of systems of particles in the plane or in space. For instance, Figure 15.1 shows the vector field determined by a wheel rotating on an axle. Notice that the velocity vectors are determined by the locations of their initial points—the farther a point is from the axle, the greater its velocity. Velocity fields are also determined by the flow of liquids through a container or by the flow of air currents around a moving object, as shown in Figure 15.2.

2. Gravitational fields are defined by Newton’s Law of Gravitation, which states that the force of attraction exerted on a particle of mass \( m_1 \) located at \((x, y, z)\) by a particle of mass \( m_2 \) located at \((0, 0, 0)\) is given by

\[
F(x, y, z) = \frac{-Gm_1 m_2}{r^2} \mathbf{u}
\]

where \( G \) is the gravitational constant and \( \mathbf{u} \) is the unit vector in the direction from the origin to \((x, y, z)\). In Figure 15.3, you can see that the gravitational force field \( \mathbf{F} \) has the properties that \( \mathbf{F}(x, y, z) \) always points toward the origin, and that the magnitude of \( \mathbf{F}(x, y, z) \) is the same at all points equidistant from the origin. A vector field with these two properties is called a central force field. Using the position vector \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) for the point \((x, y, z)\), you can write the gravitational field \( \mathbf{F} \) as

\[
F(x, y, z) = \frac{-Gm_1 m_2}{\| \mathbf{r} \|^2} \left( \frac{\mathbf{r}}{\| \mathbf{r} \|} \right) = \frac{-Gm_1 m_2}{\| \mathbf{r} \|^2} \mathbf{u}.
\]

3. Electric force fields are defined by Coulomb’s Law, which states that the force exerted on a particle with electric charge \( q_1 \) located at \((x, y, z)\) by a particle with electric charge \( q_2 \) located at \((0, 0, 0)\) is given by

\[
F(x, y, z) = \frac{cq_1 q_2}{\| \mathbf{r} \|} \mathbf{u}
\]

where \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \), \( \mathbf{u} = \mathbf{r}/\| \mathbf{r} \| \), and \( c \) is a constant that depends on the choice of units for \( \| \mathbf{r} \| \), \( q_1 \), and \( q_2 \).

Note that an electric force field has the same form as a gravitational field. That is,

\[
F(x, y, z) = \frac{k}{\| \mathbf{r} \|^2} \mathbf{u}.
\]

Such a force field is called an inverse square field.

### Definition of Inverse Square Field

Let \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \) be a position vector. The vector field \( \mathbf{F} \) is an inverse square field if

\[
F(x, y, z) = \frac{k}{\| \mathbf{r} \|^2} \mathbf{u}
\]

where \( k \) is a real number and \( \mathbf{u} = \mathbf{r}/\| \mathbf{r} \| \) is a unit vector in the direction of \( \mathbf{r} \).
Because vector fields consist of infinitely many vectors, it is not possible to create a sketch of the entire field. Instead, when you sketch a vector field, your goal is to sketch representative vectors that help you visualize the field.

**EXAMPLE 1**  **Sketching a Vector Field**

Sketch some vectors in the vector field given by

\[ \mathbf{F}(x, y) = -yi + xj. \]

**Solution**  You could plot vectors at several random points in the plane. However, it is more enlightening to plot vectors of equal magnitude. This corresponds to finding level curves in scalar fields. In this case, vectors of equal magnitude lie on circles.

\[
\begin{align*}
\|\mathbf{F}\| &= c & \text{Vectors of length } c \\
\sqrt{x^2 + y^2} &= c & \text{Equation of circle} \\
x^2 + y^2 &= c^2
\end{align*}
\]

To begin making the sketch, choose a value for \( c \) and plot several vectors on the resulting circle. For instance, the following vectors occur on the unit circle.

<table>
<thead>
<tr>
<th>Point</th>
<th>Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0)</td>
<td>( \mathbf{F}(1, 0) = \mathbf{j} )</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>( \mathbf{F}(0, 1) = -\mathbf{i} )</td>
</tr>
<tr>
<td>(-1, 0)</td>
<td>( \mathbf{F}(-1, 0) = -\mathbf{j} )</td>
</tr>
<tr>
<td>(0, -1)</td>
<td>( \mathbf{F}(0, -1) = \mathbf{i} )</td>
</tr>
</tbody>
</table>

These and several other vectors in the vector field are shown in Figure 15.4. Note in the figure that this vector field is similar to that given by the rotating wheel shown in Figure 15.1.

**EXAMPLE 2**  **Sketching a Vector Field**

Sketch some vectors in the vector field given by

\[ \mathbf{F}(x, y) = 2xi + yj. \]

**Solution**  For this vector field, vectors of equal length lie on ellipses given by

\[
\|\mathbf{F}\| = \sqrt{(2x)^2 + (y)^2} = c
\]

which implies that

\[ 4x^2 + y^2 = c^2. \]

For \( c = 1 \), sketch several vectors \( 2xi + yj \) of magnitude 1 at points on the ellipse given by

\[ 4x^2 + y^2 = 1. \]

For \( c = 2 \), sketch several vectors \( 2xi + yj \) of magnitude 2 at points on the ellipse given by

\[ 4x^2 + y^2 = 4. \]

These vectors are shown in Figure 15.5.
EXAMPLE 3  Sketching a Velocity Field

Sketch some vectors in the velocity field given by
\[ \mathbf{v}(x, y, z) = (16 - x^2 - y^2)\mathbf{k} \]
where \( x^2 + y^2 \leq 16. \)

Solution  You can imagine that \( \mathbf{v} \) describes the velocity of a liquid flowing through a tube of radius 4. Vectors near the \( z \)-axis are longer than those near the edge of the tube. For instance, at the point \((0, 0, 0)\), the velocity vector is \( \mathbf{v}(0, 0, 0) = 16\mathbf{k} \), whereas at the point \((0, 3, 0)\), the velocity vector is \( \mathbf{v}(0, 3, 0) = 7\mathbf{k} \). Figure 15.6 shows these and several other vectors for the velocity field. From the figure, you can see that the speed of the liquid is greater near the center of the tube than near the edges of the tube.

Conservative Vector Fields

Notice in Figure 15.5 that all the vectors appear to be normal to the level curve from which they emanate. Because this is a property of gradients, it is natural to ask whether the vector field given by \( \mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j} \) is the gradient for some differentiable function \( f \). The answer is that some vector fields can be represented as the gradients of differentiable functions and some cannot—those that can are called conservative vector fields.

EXAMPLE 4  Conservative Vector Fields

a. The vector field given by \( \mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j} \) is conservative. To see this, consider the potential function \( f(x, y, z) = x^2 + \frac{1}{2}y^2 \). Because
\[ \nabla f = 2x\mathbf{i} + y\mathbf{j} = \mathbf{F} \]
it follows that \( \mathbf{F} \) is conservative.

b. Every inverse square field is conservative. To see this, let
\[ \mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^3} \mathbf{u} \quad \text{and} \quad f(x, y, z) = -k \frac{1}{\sqrt{x^2 + y^2 + z^2}} \]
where \( \mathbf{u} = \mathbf{r}/\|\mathbf{r}\| \). Because
\begin{align*}
\nabla f &= \frac{kx}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{ky}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{kz}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k} \\
&= \frac{k}{x^2 + y^2 + z^2} \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \mathbf{k} \right) \\
&= \frac{k}{\|\mathbf{r}\|^2} \mathbf{r} \\
&= \frac{k}{\|\mathbf{r}\|^2} \mathbf{u}
\end{align*}
it follows that \( \mathbf{F} \) is conservative.
As can be seen in Example 4(b), many important vector fields, including gravitational fields and electric force fields, are conservative. Most of the terminology in this chapter comes from physics. For example, the term “conservative” is derived from the classic physical law regarding the conservation of energy. This law states that the sum of the kinetic energy and the potential energy of a particle moving in a conservative force field is constant. (The kinetic energy of a particle is the energy due to its motion, and the potential energy is the energy due to its position in the force field.)

The following important theorem gives a necessary and sufficient condition for a vector field in the plane to be conservative.

**THEOREM 15.1 Test for Conservative Vector Field in the Plane**

Let $M$ and $N$ have continuous first partial derivatives on an open disk $R$. The vector field given by $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$ 

**Proof** To prove that the given condition is necessary for $\mathbf{F}$ to be conservative, suppose there exists a potential function $f$ such that

$$\mathbf{F}(x, y) = \nabla f(x, y) = M\mathbf{i} + N\mathbf{j}.$$ 

Then you have

$$f_x(x, y) = M \quad \Rightarrow \quad f_y(x, y) = \frac{\partial M}{\partial y}$$

$$f_y(x, y) = N \quad \Rightarrow \quad f_x(x, y) = \frac{\partial N}{\partial x}$$

and, by the equivalence of the mixed partials $f_{xy}$ and $f_{yx}$, you can conclude that $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ for all $(x, y)$ in $R$. The sufficiency of the condition is proved in Section 15.4.

**NOTE** Theorem 15.1 requires that the domain of $\mathbf{F}$ be an open disk. If $R$ is simply an open region, the given condition is necessary but not sufficient to produce a conservative vector field.

**EXAMPLE 5 Testing for Conservative Vector Field in the Plane**

Decide whether the vector field given by $\mathbf{F}$ is conservative.

a. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$  
   b. $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

**Solution**

a. The vector field given by $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$ is not conservative because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[x^2] = x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[xy] = y.$$ 

b. The vector field given by $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$ is conservative because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[2x] = 0 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[y] = 0.$$ 

**Try It**  **Exploration A**
Theorem 15.1 tells you whether a vector field is conservative. It does not tell you how to find a potential function of \( \mathbf{F} \). The problem is comparable to antidifferentiation. Sometimes you will be able to find a potential function by simple inspection. For instance, in Example 4 you observed that

\[
f(x, y) = x^2 + \frac{1}{2} y^2
\]

has the property that \( \nabla f(x, y) = 2x \mathbf{i} + y \mathbf{j} \).

**Example 6  Finding a Potential Function for \( \mathbf{F}(x, y) \)**

Find a potential function for

\[
\mathbf{F}(x, y) = 2xy \mathbf{i} + (x^2 - y)\mathbf{j}
\]

**Solution**  From Theorem 15.1 it follows that \( \mathbf{F} \) is conservative because

\[
\frac{\partial}{\partial y}[2xy] = 2x \quad \text{and} \quad \frac{\partial}{\partial x}[x^2 - y] = 2x.
\]

If \( f \) is a function whose gradient is equal to \( \mathbf{F}(x, y) \), then

\[
\nabla f(x, y) = 2xy \mathbf{i} + (x^2 - y)\mathbf{j}
\]

which implies that

\[
f_x(x, y) = 2xy
\]

and

\[
f_y(x, y) = x^2 - y.
\]

To reconstruct the function \( f \) from these two partial derivatives, integrate \( f_x(x, y) \) with respect to \( x \) and \( f_y(x, y) \) with respect to \( y \), as follows.

\[
f(x, y) = \int f_x(x, y) \, dx = \int 2xy \, dx = x^2y + g(y)
\]

\[
f(x, y) = \int f_y(x, y) \, dy = \int (x^2 - y) \, dy = x^2y - \frac{y^2}{2} + h(x)
\]

Notice that \( g(y) \) is constant with respect to \( x \) and \( h(x) \) is constant with respect to \( y \). To find a single expression that represents \( f(x, y) \), let

\[
g(y) = -\frac{y^2}{2} \quad \text{and} \quad h(x) = K.
\]

Then, you can write

\[
f(x, y) = x^2y + g(y) + K
\]

\[
= x^2y - \frac{y^2}{2} + K.
\]

You can check this result by forming the gradient of \( f \). You will see that it is equal to the original function \( \mathbf{F} \).

**NOTE**  Notice that the solution in Example 6 is comparable to that given by an indefinite integral. That is, the solution represents a family of potential functions, any two of which differ by a constant. To find a unique solution, you would have to be given an initial condition satisfied by the potential function.
Curl of a Vector Field

Theorem 15.1 has a counterpart for vector fields in space. Before stating that result, the definition of the curl of a vector field in space is given.

**Definition of Curl of a Vector Field**

The curl of \( \mathbf{F}(x, y, z) = Mi + Nj + Pk \) is

\[
\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z) = \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left( \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.
\]

**NOTE** If \( \text{curl } \mathbf{F} = \mathbf{0} \), then \( \mathbf{F} \) is said to be irrotational.

The cross product notation used for curl comes from viewing the gradient \( \nabla f \) as the result of the differential operator \( \nabla \) acting on the function \( f \). In this context, you can use the following determinant form as an aid in remembering the formula for curl.

\[
\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \begin{vmatrix} \frac{\partial P}{\partial y} & \frac{\partial N}{\partial z} \\ \frac{\partial P}{\partial x} & \frac{\partial M}{\partial z} \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial P}{\partial x} & \frac{\partial M}{\partial z} \\ \frac{\partial N}{\partial x} & \frac{\partial M}{\partial y} \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial N}{\partial x} & \frac{\partial M}{\partial y} \\ \frac{\partial P}{\partial x} & \frac{\partial M}{\partial z} \end{vmatrix} \mathbf{k}.
\]

**EXAMPLE 7 Finding the Curl of a Vector Field**

Find \( \text{curl } \mathbf{F} \) for the vector field given by

\[
\mathbf{F}(x, y, z) = 2xy \mathbf{i} + (x^2 + z^2) \mathbf{j} + 2yz \mathbf{k}.
\]

Is \( \mathbf{F} \) irrotational?

**Solution** The curl of \( \mathbf{F} \) is given by

\[
\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 + z^2 & 2yz \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \end{vmatrix} \mathbf{k} = (2z - 2z) \mathbf{i} - (0 - 0) \mathbf{j} + (2x - 2x) \mathbf{k} = 0.
\]

Because \( \text{curl } \mathbf{F} = \mathbf{0} \), \( \mathbf{F} \) is irrotational.
Later in this chapter, you will assign a physical interpretation to the curl of a vector field. But for now, the primary use of curl is shown in the following test for conservative vector fields in space. The test states that for a vector field whose domain is all of three-dimensional space (or an open sphere), the curl is \( \mathbf{0} \) at every point in the domain if and only if \( \mathbf{F} \) is conservative. The proof is similar to that given for Theorem 15.1.

**THEOREM 15.2 Test for Conservative Vector Field in Space**

Suppose that \( M, N, \) and \( P \) have continuous first partial derivatives in an open sphere \( Q \) in space. The vector field given by \( \mathbf{F}(x, y, z) = Mi + Nj +Pk \) is conservative if and only if

\[
\text{curl} \, \mathbf{F}(x, y, z) = \mathbf{0}.
\]

That is, \( \mathbf{F} \) is conservative if and only if

\[
\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} = \frac{\partial P}{\partial z} - \frac{\partial M}{\partial y} = \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}.
\]

From Theorem 15.2, you can see that the vector field given in Example 7 is conservative because \( \text{curl} \, \mathbf{F}(x, y, z) = \mathbf{0} \). Try showing that the vector field

\[
\mathbf{F}(x, y, z) = x^3y^2zi + x^2zj + x^2yk
\]

is not conservative—you can do this by showing that its curl is

\[
\text{curl} \, \mathbf{F}(x, y, z) = (x^3y^2 - 2xy)j + (2xz - 2x^3yz)k \neq \mathbf{0}.
\]

For vector fields in space that pass the test for being conservative, you can find a potential function by following the same pattern used in the plane (as demonstrated in Example 6).

**EXAMPLE 8 Finding a Potential Function for \( \mathbf{F}(x, y, z) \)**

Find a potential function for \( \mathbf{F}(x, y, z) = 2xyi + (x^2 + z^2)j + 2yzk \).

**Solution** From Example 7, you know that the vector field given by \( \mathbf{F} \) is conservative. If \( f \) is a function such that \( \mathbf{F}(x, y, z) = \nabla f(x, y, z) \), then

\[
f_x(x, y, z) = 2xy, \quad f_y(x, y, z) = x^2 + z^2, \quad \text{and} \quad f_z(x, y, z) = 2yz
\]

and integrating with respect to \( x, y, \) and \( z \) separately produces

\[
f(x, y, z) = \int M \, dx = \int 2xy \, dx = x^2y + g(y, z)
\]

\[
f(x, y, z) = \int N \, dy = \int (x^2 + z^2) \, dy = x^2y + yz^2 + h(x, z)
\]

\[
f(x, y, z) = \int P \, dz = \int 2yz \, dz = yz^2 + k(x, y).
\]

Comparing these three versions of \( f(x, y, z) \), you can conclude that

\[
g(y, z) = yz^2 + K, \quad h(x, z) = K, \quad \text{and} \quad k(x, y) = x^2y + K.
\]

So, \( f(x, y, z) \) is given by

\[
f(x, y, z) = x^2y + yz^2 + K.
\]
Divergence of a Vector Field

You have seen that the curl of a vector field \( \mathbf{F} \) is itself a vector field. Another important function defined on a vector field is divergence, which is a scalar function.

**Definition of Divergence of a Vector Field**

The divergence of \( \mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j} \) is

\[
\text{div } \mathbf{F}(x, y) = \nabla \cdot \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.
\]

Plane

The divergence of \( \mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) is

\[
\text{div } \mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.
\]

Space

If \( \text{div } \mathbf{F} = 0 \), then \( \mathbf{F} \) is said to be divergence free.

The dot product notation used for divergence comes from considering \( \nabla \) as a differential operator, as follows.

\[
\nabla \cdot \mathbf{F}(x, y, z) = \left( \frac{\partial}{\partial x} \right) \mathbf{i} + \left( \frac{\partial}{\partial y} \right) \mathbf{j} + \left( \frac{\partial}{\partial z} \right) \mathbf{k} \cdot (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.
\]

**EXAMPLE 9** Finding the Divergence of a Vector Field

Find the divergence at \((2, 1, -1)\) for the vector field

\[
\mathbf{F}(x, y, z) = x^3y^2\mathbf{i} + x^2z\mathbf{j} + x^3y\mathbf{k}.
\]

**Solution** The divergence of \( \mathbf{F} \) is

\[
\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[x^3y^2] + \frac{\partial}{\partial y}[x^2z] + \frac{\partial}{\partial z}[x^3y] = 3x^3y^2 + 2x^2.
\]

At the point \((2, 1, -1)\), the divergence is

\[
\text{div } \mathbf{F}(2, 1, -1) = 3(2^3)(1^2)(-1) = -12.
\]

**Try It**

There are many important properties of the divergence and curl of a vector field \( \mathbf{F} \) (see Exercises 77–83). One that is used often is described in Theorem 15.3. You are asked to prove this theorem in Exercise 84.

**THEOREM 15.3** Relationship Between Divergence and Curl

If \( \mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) is a vector field and \( M, N, \) and \( P \) have continuous second partial derivatives, then

\[
\text{div } (\text{curl } \mathbf{F}) = 0.
\]
Exercises for Section 15.1

In Exercises 1–6, match the vector field with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a) \[ F(x, y) = x \hat{i} \]  
(b) \[ F(x, y) = y \hat{i} \]  
(c) \[ F(x, y) = x \hat{i} + 3y \hat{j} \]  
(d) \[ F(x, y) = y \hat{i} - x \hat{j} \]  
(e) \[ F(x, y) = (x, \sin y) \]  
(f) \[ F(x, y) = (\frac{1}{3}xy, \frac{1}{3}z^2) \]

1. \[ F(x, y) = x \hat{j} \]  
2. \[ F(x, y) = y \hat{i} \]  
3. \[ F(x, y) = x \hat{i} + 3y \hat{j} \]  
4. \[ F(x, y) = y \hat{i} - x \hat{j} \]  
5. \[ F(x, y) = (x, \sin y) \]  
6. \[ F(x, y) = (\frac{1}{3}xy, \frac{1}{3}z^2) \]

In Exercises 7–16, sketch several representative vectors in the vector field.

7. \[ F(x, y) = \hat{i} + \hat{j} \]  
8. \[ F(x, y) = 2\hat{i} \]  
9. \[ F(x, y) = x \hat{i} + y \hat{j} \]  
10. \[ F(x, y) = x \hat{i} - y \hat{j} \]  
11. \[ F(x, y, z) = 3\hat{j} \]  
12. \[ F(x, y, z) = \hat{x} \]  
13. \[ F(x, y, z) = 4x \hat{i} + y \hat{j} \]  
14. \[ F(x, y, z) = (x^2 + y^2)\hat{i} + \hat{j} \]  
15. \[ F(x, y, z) = \hat{i} + \hat{j} + \hat{k} \]  
16. \[ F(x, y, z) = x \hat{i} + y \hat{j} + z \hat{k} \]

In Exercises 17–20, use a computer algebra system to graph several representative vectors in the vector field.

17. \[ F(x, y) = \frac{1}{2}(2xy \hat{i} + y^2 \hat{j}) \]  
18. \[ F(x, y) = (2y - 3x)\hat{i} + (2y + 3x)\hat{j} \]  
19. \[ F(x, y, z) = \frac{x \hat{i} + y \hat{j} + z \hat{k}}{\sqrt{x^2 + y^2 + z^2}} \]  
20. \[ F(x, y, z) = x \hat{i} - y \hat{j} + z \hat{k} \]

In Exercises 21–26, find the gradient vector field for the scalar function. (That is, find the conservative vector field for the potential function.)

21. \[ f(x, y) = 5x^2 + 3xy + 10y^2 \]  
22. \[ f(x, y) = \sin 3x \cos 4y \]  
23. \[ f(x, y, z) = z - ye^{x^2} \]  
24. \[ f(x, y, z) = \frac{y + z - x^2}{x} \]  
25. \[ g(x, y, z) = xy \ln(x + y) \]  
26. \[ g(x, y, z) = x \arcsin yz \]

In Exercises 27–30, verify that the vector field is conservative.

27. \[ F(x, y) = 12xy \hat{i} + 6x^2 + y \hat{j} \]  
28. \[ F(x, y) = \frac{1}{x^2}(y \hat{i} - x \hat{j}) \]  
29. \[ F(x, y) = \sin y \hat{i} + x \cos y \hat{j} \]  
30. \[ F(x, y) = \frac{1}{xy}(y \hat{i} - x \hat{j}) \]

In Exercises 31–34, determine if the vector field is conservative.

31. \[ F(x, y) = 5y^3(3y \hat{i} - x \hat{j}) \]  
32. \[ F(x, y) = \frac{1}{\sqrt{x^2 + y^2}}(x \hat{i} + y \hat{j}) \]  
33. \[ F(x, y) = \frac{2}{y^2}e^{2x/y}(y \hat{i} - x \hat{j}) \]  
34. \[ F(x, y) = \frac{1}{\sqrt{1-x^2-y^2}}(y \hat{i} - x \hat{j}) \]

In Exercises 35–42, determine whether the vector field is conservative. If it is, find a potential function for the vector field.

35. \[ F(x, y) = 2xy \hat{i} + x^2 \hat{j} \]  
36. \[ F(x, y) = \frac{1}{y^2}(y \hat{i} - 2x \hat{j}) \]  
37. \[ F(x, y) = xe^{y^2}(2y \hat{i} + x \hat{j}) \]  
38. \[ F(x, y) = 3x^2y \hat{i} + 2x^3y \hat{j} \]  
39. \[ F(x, y) = \frac{x \hat{i} + y \hat{j}}{x^2 + y^2} \]  
40. \[ F(x, y) = \frac{2y}{x} \hat{i} - \frac{x^2}{y^2} \hat{j} \]  
41. \[ F(x, y) = e^y(\cos y \hat{i} + \sin y \hat{j}) \]  
42. \[ F(x, y) = \frac{2x \hat{i} + 2y \hat{j}}{(x^2 + y^2)^2} \]

In Exercises 43–46, find \( \text{curl} \, F \) for the vector field at the given point.

<table>
<thead>
<tr>
<th>Vector Field</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(x, y, z) = xy \hat{i} + y \hat{j} + z \hat{k} )</td>
<td>(1, 2, 1)</td>
</tr>
<tr>
<td>( F(x, y, z) = x^2 \hat{i} + 2x \hat{j} - 2y \hat{k} )</td>
<td>(2, -1, 3)</td>
</tr>
<tr>
<td>( F(x, y, z) = e^{-xyz}(x \hat{i} + j \hat{j} + k \hat{k}) )</td>
<td>(0, 0, 3)</td>
</tr>
<tr>
<td>( F(x, y, z) = e^{-xyz}(x \hat{i} + j \hat{j} + k \hat{k}) )</td>
<td>(3, 2, 0)</td>
</tr>
</tbody>
</table>
In Exercises 47–50, use a computer algebra system to find the curl \( \mathbf{F} \) for the vector field.

47. \( \mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right) \mathbf{i} + \ln \sqrt{x^2 + y^2} \mathbf{j} + \mathbf{k} \)

48. \( \mathbf{F}(x, y, z) = \frac{yz}{y - z} \mathbf{i} + \frac{xz}{x - z} \mathbf{j} + \frac{xy}{x - y} \mathbf{k} \)

49. \( \mathbf{F}(x, y, z) = \sin(x - y) \mathbf{i} + \sin(y - z) \mathbf{j} + \sin(z - x) \mathbf{k} \)

50. \( \mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2} (i + j + k) \)

In Exercises 51–56, determine whether the vector field \( \mathbf{F} \) is conservative. If it is, find a potential function for the vector field.

51. \( \mathbf{F}(x, y, z) = \sin y \mathbf{i} - x \cos y \mathbf{j} + \mathbf{k} \)

52. \( \mathbf{F}(x, y, z) = e^z (y \mathbf{i} + x \mathbf{j} + xy \mathbf{k}) \)

53. \( \mathbf{F}(x, y, z) = e^z (y \mathbf{i} + x \mathbf{j} + xy \mathbf{k}) \)

54. \( \mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^2 \mathbf{j} + 3xy^2 z^2 \mathbf{k} \)

55. \( \mathbf{F}(x, y, z) = \frac{1}{y} \mathbf{i} - \frac{x}{y} \mathbf{j} + (2z - 1) \mathbf{k} \)

56. \( \mathbf{F}(x, y, z) = \frac{x}{x^2 + y^2} \mathbf{i} + \frac{y}{x^2 + y^2} \mathbf{j} + \mathbf{k} \)

In Exercises 57–60, find the divergence of the vector field \( \mathbf{F} \).

57. \( \mathbf{F}(x, y, z) = 6x^2 \mathbf{i} - xy^2 \mathbf{j} \)

58. \( \mathbf{F}(x, y, z) = xe^z \mathbf{i} + ye^z \mathbf{j} \)

59. \( \mathbf{F}(x, y, z) = \sin x \mathbf{i} + \cos y \mathbf{j} + z^2 \mathbf{k} \)

60. \( \mathbf{F}(x, y, z) = \ln(x^2 + y^2) \mathbf{i} + xy \mathbf{j} + \ln(y^2 + z^2) \mathbf{k} \)

In Exercises 61–64, find the divergence of the vector field \( \mathbf{F} \) at the given point.

<table>
<thead>
<tr>
<th>Vector Field</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{F}(x, y, z) = xyz \mathbf{i} + yz \mathbf{j} + zk )</td>
<td>( (1, 2, 1) )</td>
</tr>
<tr>
<td>( \mathbf{F}(x, y, z) = x^2 \mathbf{i} - 2xz \mathbf{j} + yzk )</td>
<td>( (2, -1, 3) )</td>
</tr>
<tr>
<td>( \mathbf{F}(x, y, z) = e^z \sin y \mathbf{i} - e^z \cos y \mathbf{j} )</td>
<td>( (0, 0, 3) )</td>
</tr>
<tr>
<td>( \mathbf{F}(x, y, z) = \ln(xyz) (i + j + k) )</td>
<td>( (3, 2, 1) )</td>
</tr>
</tbody>
</table>

In Exercises 69 and 70, find \( \text{curl} (\mathbf{F} \times \mathbf{G}) \).

69. \( \mathbf{F}(x, y, z) = \mathbf{i} + 2xz \mathbf{j} + 3yk \)
   \( \mathbf{G}(x, y, z) = xi - yz \mathbf{k} \)

70. \( \mathbf{F}(x, y, z) = xi - zk \)
   \( \mathbf{G}(x, y, z) = x^2 i + yj + zk \)

In Exercises 71 and 72, find \( \text{curl} (\text{curl} \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) \).

71. \( \mathbf{F}(x, y, z) = xzy \mathbf{i} + yz \mathbf{j} + zk \)

72. \( \mathbf{F}(x, y, z) = x^2 zi - 2zxj + yzk \)

In Exercises 73 and 74, find \( \text{div} (\mathbf{F} \times \mathbf{G}) \).

73. \( \mathbf{F}(x, y, z) = i + 2xz \mathbf{j} + 3yk \)
   \( \mathbf{G}(x, y, z) = xi - zk \)

74. \( \mathbf{F}(x, y, z) = xi - zk \)
   \( \mathbf{G}(x, y, z) = x^2i + yj + zk \)

In Exercises 75 and 76, find \( \text{div} (\text{curl} \mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) \).

75. \( \mathbf{F}(x, y, z) = xzy \mathbf{i} + yz \mathbf{j} + zk \)

76. \( \mathbf{F}(x, y, z) = x^2zi - 2zxj + yzk \)

In Exercises 77–84, prove the property for vector fields \( \mathbf{F} \) and \( \mathbf{G} \) and scalar function \( f \). (Assume that the required partia derivatives are continuous.)

77. \( \text{curl} (\mathbf{F} + \mathbf{G}) = \text{curl} \mathbf{F} + \text{curl} \mathbf{G} \)

78. \( \text{curl}(\nabla f) = \nabla \times (\nabla f) = 0 \)

79. \( \text{div}(\mathbf{F} + \mathbf{G}) = \text{div} \mathbf{F} + \text{div} \mathbf{G} \)

80. \( \text{div} (\mathbf{F} \times \mathbf{G}) = (\text{curl} \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\text{curl} \mathbf{G}) \)

81. \( \nabla \times (\nabla f + \nabla G) = \nabla \times (\nabla f) + \nabla \times (\nabla G) \)

82. \( \nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) \)

83. \( \text{div}(\mathbf{f} \cdot \mathbf{G}) = f \text{div} \mathbf{G} + \mathbf{G} \cdot \nabla f \cdot \mathbf{F} \)

84. \( \text{div}(\text{curl} \mathbf{F}) = 0 \) (Theorem 15.3)

In Exercises 85–88, let \( \mathbf{F}(x, y, z) = xi + yj + zk \) and let \( f(x, y, z) = \|\mathbf{F}(x, y, z)\| \).

85. Show that \( \nabla (\ln f) = \frac{\mathbf{F}}{f^2} \)

86. Show that \( \nabla \left( \frac{1}{f} \right) = -\frac{\mathbf{F}}{f^3} \)

87. Show that \( \nabla f^n = nf^{n - 1} \mathbf{F} \).

88. The Laplacian is the differential operator

\[
\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

and Laplace’s equation is

\[
\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0.
\]

Any function that satisfies this equation is called harmonic. Show that the function \( 1/f \) is harmonic.

**True or False?** In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

89. If \( \mathbf{F}(x, y) = 4xi - y^2 \mathbf{j} \), then \( \|\mathbf{F}(x, y)\| \rightarrow 0 \) as \( (x, y) \rightarrow (0, 0) \).

90. If \( \mathbf{F}(x, y) = 4xi - y^2 \mathbf{j} \) and \( (x, y) \) is on the positive y-axis, then the vector points in the negative y-direction.

91. If \( f \) is a scalar field, then \( \text{curl} \ f \) is a meaningful expression.

92. If \( \mathbf{F} \) is a vector field and \( \text{curl} \ \mathbf{F} = 0 \), then \( \mathbf{F} \) is irrotational but not conservative.


**Section 15.2**  

**Line Integrals**

- Understand and use the concept of a piecewise smooth curve.
- Write and evaluate a line integral.
- Write and evaluate a line integral of a vector field.
- Write and evaluate a line integral in differential form.

**Piecewise Smooth Curves**

A classic property of gravitational fields is that, subject to certain physical constraints, the work done by gravity on an object moving between two points in the field is independent of the path taken by the object. One of the constraints is that the path must be a piecewise smooth curve. Recall that a plane curve $C$ given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \leq t \leq b$$

is smooth if

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}$$

are continuous on $[a, b]$ and not simultaneously 0 on $(a, b)$. Similarly, a space curve $C$ given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b$$

is smooth if

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \text{and} \quad \frac{dz}{dt}$$

are continuous on $[a, b]$ and not simultaneously 0 on $(a, b)$. A curve $C$ is piecewise smooth if the interval $[a, b]$ can be partitioned into a finite number of subintervals, on each of which $C$ is smooth.

**EXAMPLE 1  Finding a Piecewise Smooth Parametrization**

Find a piecewise smooth parametrization of the graph of $C$ shown in Figure 15.7.

**Solution**  Because $C$ consists of three line segments $C_1$, $C_2$, and $C_3$, you can construct a smooth parametrization for each segment and piece them together by making the last $t$-value in $C_i$ correspond to the first $t$-value in $C_{i+1}$, as follows.

\[
C_1: \quad x(t) = 0, \quad y(t) = 2t, \quad z(t) = 0, \quad 0 \leq t \leq 1 \\
C_2: \quad x(t) = t - 1, \quad y(t) = 2, \quad z(t) = 0, \quad 1 \leq t \leq 2 \\
C_3: \quad x(t) = 1, \quad y(t) = 2, \quad z(t) = t - 2, \quad 2 \leq t \leq 3
\]

So, $C$ is given by

$$\mathbf{r}(t) = \begin{cases} 
2t\mathbf{j}, & 0 \leq t \leq 1 \\
(t - 1)\mathbf{i} + 2\mathbf{j}, & 1 \leq t \leq 2 \\
\mathbf{i} + 2\mathbf{j} + (t - 2)\mathbf{k}, & 2 \leq t \leq 3
\end{cases}$$

Because $C_1$, $C_2$, and $C_3$ are smooth, it follows that $C$ is piecewise smooth.

**Try It**  **Exploration A**

Recall that parametrization of a curve induces an orientation to the curve. For instance, in Example 1, the curve is oriented such that the positive direction is from $(0, 0, 0)$, following the curve to $(1, 2, 1)$. Try finding a parametrization that induces the opposite orientation.
**Line Integrals**

Up to this point in the text, you have studied various types of integrals. For a single integral

\[ \int_{a}^{b} f(x) \, dx \]

you integrated over the interval \([a, b]\). Similarly, for a double integral

\[ \int_{R} f(x, y) \, dA \]

you integrated over the region \(R\) in the plane. In this section, you will study a new type of integral called a **line integral**

\[ \int_{C} f(x, y) \, ds \]

for which you integrate over a piecewise smooth curve \(C\). (The terminology is somewhat unfortunate—the type of integral might be better described as a “curve integral.”)

To introduce the concept of a line integral, consider the mass of a wire of finite length, given by a curve in space. The density (mass per unit length) of the wire at the point \((x, y, z)\) is given by \(f(x, y, z)\). Partition the curve \(C\) by the points

\[ P_0, P_1, \ldots, P_n \]

producing \(n\) subarcs, as shown in Figure 15.8. The length of the \(i\)th subarc is given by \(\Delta s_i\). Next, choose a point \((x_i, y_i, z_i)\) in each subarc. If the length of each subarc is small, the total mass of the wire can be approximated by the sum

\[
\text{Mass of wire} \approx \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta s_i.
\]

If you let \(\|\Delta\|\) denote the length of the longest subarc and let \(\|\Delta\|\) approach 0, it seems reasonable that the limit of this sum approaches the mass of the wire. This leads to the following definition.

**Definition of Line Integral**

If \(f\) is defined in a region containing a smooth curve \(C\) of finite length, then the **line integral of \(f\ along \(C\)** is given by

\[
\int_{C} f(x, y) \, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta s_i \quad \text{Plane}
\]

or

\[
\int_{C} f(x, y, z) \, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta s_i \quad \text{Space}
\]

provided this limit exists.

As with the integrals discussed in Chapter 14, evaluation of a line integral is best accomplished by converting to a definite integral. It can be shown that if \(f\) is **continuous**, the limit given above exists and is the same for all smooth parametrizations of \(C\).
To evaluate a line integral over a plane curve $C$ given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, use the fact that
\[ ds = \|\mathbf{r}'(t)\| \, dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt. \]
A similar formula holds for a space curve, as indicated in Theorem 15.4.

**THEOREM 15.4 Evaluation of a Line Integral as a Definite Integral**

Let $f$ be continuous in a region containing a smooth curve $C$. If $C$ is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \leq t \leq b$, then
\[ \int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt. \]

If $C$ is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \leq t \leq b$, then
\[ \int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt. \]

Note that if $f(x, y, z) = 1$, the line integral gives the arc length of the curve $C$, as defined in Section 12.5. That is,
\[ \int_C 1 \, ds = \int_a^b \|\mathbf{r}'(t)\| \, dt = \text{length of curve } C. \]

**EXAMPLE 2 Evaluating a Line Integral**

Evaluate
\[ \int_C (x^2 - y + 3z) \, ds \]
where $C$ is the line segment shown in Figure 15.9.

**Solution** Begin by writing a parametric form of the equation of a line:
\[ x = t, \quad y = 2t, \quad z = t, \quad 0 \leq t \leq 1. \]

Therefore, $x'(t) = 1$, $y'(t) = 2$, and $z'(t) = 1$, which implies that
\[ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}. \]

So, the line integral takes the following form.
\[ \int_C (x^2 - y + 3z) \, ds = \int_0^1 (t^2 - 2t + 3t) \sqrt{6} \, dt \]
\[ = \sqrt{6} \int_0^1 (t^2 + t) \, dt \]
\[ = \sqrt{6} \left[ \frac{t^3}{3} + \frac{t^2}{2} \right]_0^1 \]
\[ = \frac{5\sqrt{6}}{6} \]

**Try It** **Exploration A**
Suppose $C$ is a path composed of smooth curves $C_1$, $C_2$, \ldots, $C_n$. If $f$ is continuous on $C$, it can be shown that
\[
\int_C f(x, y) \, ds = \int_{C_1} f(x, y) \, ds + \int_{C_2} f(x, y) \, ds + \cdots + \int_{C_n} f(x, y) \, ds.
\]
This property is used in Example 3.

**EXAMPLE 3  Evaluating a Line Integral Over a Path**

Evaluate $\int_C x \, ds$, where $C$ is the piecewise smooth curve shown in Figure 15.10.

**Solution**  Begin by integrating up the line $y = x$, using the following parametrization.

For this curve, $r(t) = t\, \mathbf{i} + t\, \mathbf{j}$, which implies that $x'(t) = 1$ and $y'(t) = 1$. So,
\[
\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{2}
\]
and you have
\[
\int_{C_1} x \, ds = \int_0^1 t \sqrt{2} \, dt = \left[ \frac{\sqrt{2}}{2} t^2 \right]_0^1 = \frac{\sqrt{2}}{2}.
\]
Next, integrate down the parabola $y = x^2$, using the parametrization

For this curve, $r(t) = (1 - t)\, \mathbf{i} + (1 - t)^2\, \mathbf{j}$, which implies that $x'(t) = -1$ and $y'(t) = -2(1 - t)$. So,
\[
\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1 + 4(1 - t)^2}
\]
and you have
\[
\int_{C_2} x \, ds = \int_0^1 (1 - t) \sqrt{1 + 4(1 - t)^2} \, dt
= -\frac{1}{8} \left[ \frac{2}{3} \left( 1 + 4(1 - t)^2 \right)^{3/2} \right]_0^1
= \frac{1}{12} (5^{3/2} - 1).
\]
Consequently,
\[
\int_C x \, ds = \int_{C_1} x \, ds + \int_{C_2} x \, ds = \frac{\sqrt{2}}{2} + \frac{1}{12} (5^{3/2} - 1) \approx 1.56.
\]

**Try It**

For parametrizations given by $r(t) = x(t)\, \mathbf{i} + y(t)\, \mathbf{j} + z(t)\, \mathbf{k}$, it is helpful to remember the form of $ds$ as
\[
ds = \|r'(t)\| \, dt = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt.
\]
This is demonstrated in Example 4.
EXAMPLE 4  Evaluating a Line Integral

Evaluate \( \int_C (x + 2) \, ds \), where \( C \) is the curve represented by

\[
r(t) = ti + \frac{4}{3} t^{3/2} j + \frac{1}{2} t^2 k, \quad 0 \leq t \leq 2.
\]

Solution  Because \( r'(t) = i + 2t^{1/2} j + rk \), and

\[
\|r'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{1 + 4t + t^2}
\]

it follows that

\[
\int_C (x + 2) \, ds = \int_0^2 (t + 2)\sqrt{1 + 4t + t^2} \, dt
\]
\[
= \frac{1}{2} \int_0^2 2(t + 2)(1 + 4t + t^2)^{1/2} \, dt
\]
\[
= \frac{1}{3} \left[ (1 + 4t + t^2)^{3/2} \right]_0^2
\]
\[
= \frac{1}{3} (13\sqrt{13} - 1)
\]
\[
= 15.29.
\]

Try It  Exploration A

The next example shows how a line integral can be used to find the mass of a spring whose density varies. In Figure 15.11, note that the density of this spring increases as the spring spirals up the \( z \)-axis.

EXAMPLE 5  Finding the Mass of a Spring

Find the mass of a spring in the shape of the circular helix

\[
r(t) = \frac{1}{\sqrt{2}}(\cos ti + \sin tj + tk), \quad 0 \leq t \leq 6\pi
\]

where the density of the spring is \( \rho(x, y, z) = 1 + z \), as shown in Figure 15.11.

Solution  Because

\[
\|r'(t)\| = \frac{1}{\sqrt{2}}\sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = 1
\]

it follows that the mass of the spring is

\[
\text{Mass} = \int_C (1 + z) \, ds = \int_0^{6\pi} \left( 1 + \frac{t}{\sqrt{2}} \right) \, dt
\]
\[
= \left[ t + \frac{t^2}{2\sqrt{2}} \right]_0^{6\pi}
\]
\[
= 6\pi \left( 1 + \frac{3\pi}{\sqrt{2}} \right)
\]
\[
\approx 144.47.
\]

The mass of the spring is approximately 144.47.
Line Integrals of Vector Fields

One of the most important physical applications of line integrals is that of finding the work done on an object moving in a force field. For example, Figure 15.12 shows an inverse square force field similar to the gravitational field of the sun. Note that the magnitude of the force along a circular path about the center is constant, whereas the magnitude of the force along a parabolic path varies from point to point.

To see how a line integral can be used to find work done in a force field, consider an object moving along a path in the field, as shown in Figure 15.13. To determine the work done by the force, you need consider only that part of the force that is acting in the same direction as that in which the object is moving (or the opposite direction). This means that at each point on C, you can consider the projection \( F \cdot T \) of the force vector \( F \) onto the unit tangent vector \( T \). On a small subarc of length \( \Delta s_i \), the increment of work is

\[
\Delta W_i = (\text{force})(\text{distance}) = \left[ F(x_i, y_i, z_i) \cdot T(x_i, y_i, z_i) \right] \Delta s_i
\]

where \( (x_i, y_i, z_i) \) is a point in the \( i \)th subarc. Consequently, the total work done is given by the following integral.

\[
W = \int_C F(x, y, z) \cdot T(x, y, z) \, ds
\]

At each point on \( C \), the force in the direction of motion is \( (F \cdot T)T \).

This line integral appears in other contexts and is the basis of the following definition of the line integral of a vector field. Note in the definition that

\[
F \cdot T \, ds = F \cdot \frac{r'(t)}{||r'(t)||} ||r'(t)|| \, dt = F \cdot r'(t) \, dt = F \cdot dr.
\]

**Definition of Line Integral of a Vector Field**

Let \( F \) be a continuous vector field defined on a smooth curve \( C \) given by \( r(t) \), \( a \leq t \leq b \). The line integral of \( F \) on \( C \) is given by

\[
\int_C F \cdot dr = \int_C F \cdot T \, ds = \int_a^b F(x(t), y(t), z(t)) \cdot r'(t) \, dt.
\]
EXAMPLE 6 Work Done by a Force

Find the work done by the force field

$$\mathbf{F}(x, y, z) = -\frac{1}{2} x \mathbf{i} - \frac{1}{2} y \mathbf{j} + \frac{1}{4} \mathbf{k}$$  \hspace{1cm} \text{Force field } \mathbf{F}$$

on a particle as it moves along the helix given by

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$$  \hspace{1cm} \text{Space curve } C$$

from the point \((1, 0, 0)\) to \((-1, 0, 3\pi)\), as shown in Figure 15.14.

Solution Because

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

it follows that \(x(t) = \cos t, y(t) = \sin t, \text{ and } z(t) = t\). So, the force field can be written as

$$\mathbf{F}(x(t), y(t), z(t)) = -\frac{1}{2} \cos t \mathbf{i} - \frac{1}{2} \sin t \mathbf{j} + \frac{1}{4} \mathbf{k}.$$  

To find the work done by the force field in moving a particle along the curve \(C\), use the fact that

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$$

to write the following.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^{3\pi} \left( -\frac{1}{2} \cos t \mathbf{i} - \frac{1}{2} \sin t \mathbf{j} + \frac{1}{4} \mathbf{k} \right) \cdot \left( -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \right) \, dt$$

$$= \int_0^{3\pi} \left( \frac{1}{2} \sin t \cos t - \frac{1}{2} \sin t \cos t + \frac{1}{4} \right) \, dt$$

$$= \frac{3\pi}{4}$$

NOTE In Example 6, note that the \(x\)- and \(y\)-components of the force field end up contributing nothing to the total work. This occurs because in this particular example the \(z\)-component of the force field is the only portion of the force that is acting in the same (or opposite) direction in which the particle is moving (see Figure 15.15).

TECHNOLOGY The computer-generated view of the force field in Example 6 shown in Figure 15.15 indicates that each vector in the force field points toward the \(z\)-axis.
For line integrals of vector functions, the orientation of the curve \( C \) is important. If the orientation of the curve is reversed, the unit tangent vector \( T(t) \) is changed to \(-T(t)\), and you obtain
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.
\]

**EXAMPLE 7** Orientation and Parametrization of a Curve

Let \( \mathbf{F}(x, y) = yi + x^2j \) and evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) for each parabolic curve shown in Figure 15.16.

a. \( C_1: \mathbf{r}_1(t) = (4 - t)i + (4t - t^2)j, \quad 0 \leq t \leq 3 \)

b. \( C_2: \mathbf{r}_2(t) = ti + (4t - t^2)j, \quad 1 \leq t \leq 4 \)

**Solution**

a. Because \( \mathbf{r}_1'(t) = -i + (4 - 2t)j \) and
\[
\mathbf{F}(x(t), y(t)) = (4t - t^2)i + (4 - t)^2j
\]
the line integral is
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^3 [(4t - t^2)i + (4 - t)^2j] \cdot [-i + (4 - 2t)j] \, dt
\]
\[
= \int_0^3 (-4t + t^2 + 64 - 64t + 20t^2 - 2t^3) \, dt
\]
\[
= \int_0^3 (-2t^3 + 21t^2 - 68t + 64) \, dt
\]
\[
= \left[ -\frac{t^4}{2} + 7t^3 - 34t^2 + 64t \right]_0^3
\]
\[
= \frac{69}{2}.
\]

b. Because \( \mathbf{r}_2'(t) = i + (4 - 2t)j \) and
\[
\mathbf{F}(x(t), y(t)) = (4t - t^2)i + t^2j
\]
the line integral is
\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^4 [(4t - t^2)i + t^2j] \cdot [i + (4 - 2t)j] \, dt
\]
\[
= \int_1^4 (4t - t^2 + 4t^2 - 2t^3) \, dt
\]
\[
= \int_1^4 (-2t^3 + 3t^2 + 4t) \, dt
\]
\[
= \left[ -\frac{t^4}{2} + t^3 + 2t^2 \right]_1^4
\]
\[
= -\frac{69}{2}.
\]

The answer in part (b) is the negative of that in part (a) because \( C_1 \) and \( C_2 \) represent opposite orientations of the same parabolic segment.
Line Integrals in Differential Form

A second commonly used form of line integrals is derived from the vector field notation used in the preceding section. If \( \mathbf{F} \) is a vector field of the form \( \mathbf{F}(x, y) = M \mathbf{i} + N \mathbf{j} \), and \( C \) is given by \( \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} \), then \( \mathbf{F} \cdot d\mathbf{r} \) is often written as \( M \, dx + N \, dy \).

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt = \int_a^b (M \mathbf{i} + N \mathbf{j}) \cdot (x'(t) \mathbf{i} + y'(t) \mathbf{j}) \, dt = \int_a^b \left( \frac{dM}{dt} \, dt + \frac{dN}{dt} \, dt \right) = \int_a^b (M \, dx + N \, dy)
\]

This differential form can be extended to three variables. The parentheses are often omitted, as follows.

\[
\int_C M \, dx + N \, dy \quad \text{and} \quad \int_C M \, dx + N \, dy + P \, dz
\]

Notice how this differential notation is used in Example 8.

**EXAMPLE 8** Evaluating a Line Integral in Differential Form

Let \( C \) be the circle of radius 3 given by

\[
\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi
\]
as shown in Figure 15.17. Evaluate the line integral

\[
\int_C y^3 \, dx + (x^3 + 3xy^2) \, dy.
\]

**Solution** Because \( x = 3 \cos t \) and \( y = 3 \sin t \), you have \( dx = -3 \sin t \, dt \) and \( dy = 3 \cos t \, dt \). So, the line integral is

\[
\int_C M \, dx + N \, dy
\]

\[
= \int_C y^3 \, dx + (x^3 + 3xy^2) \, dy
\]

\[
= \int_0^{2\pi} [(27 \sin^3 t)(-3 \sin t) + (27 \cos^3 t + 81 \cos t \sin^2 t)(3 \cos t)] \, dt
\]

\[
= 27 \int_0^{2\pi} \sin^4 t \, dt + 3 \cos^2 t \sin^2 t \, dt
\]

\[
= 27 \int_0^{2\pi} \cos^4 t - \sin^4 t + 3 \cos^2 t \sin^2 t \, dt
\]

\[
= 27 \int_0^{2\pi} \left( \cos^2 t - \sin^2 t + 3 \cos^2 t \sin^2 t \right) \, dt
\]

\[
= 81 \int_0^{2\pi} \left( \cos^2 t + 3 \cos^2 t \sin^2 t \right) \, dt
\]

\[
= 81 \int_0^{2\pi} \sin 2t + \frac{3}{8} \left( 1 - \cos 4t \right) \, dt
\]

\[
= 81 \left[ \frac{\sin 2t}{2} + \frac{3}{8} \left( \frac{1 - \cos 4t}{4} \right) \right]_0^{2\pi}
\]

\[
= \frac{243\pi}{4}.
\]

**Try It**

**Exploration A**
For curves represented by \( y = g(x), \ a \leq x \leq b \), you can let \( x = t \) and obtain the parametric form

\[
x = t \quad \text{and} \quad y = g(t), \ a \leq t \leq b.
\]

Because \( dx = dt \) for this form, you have the option of evaluating the line integral in the variable \( x \) or \( t \). This is demonstrated in Example 9.

**EXAMPLE 9** Evaluating a Line Integral in Differential Form

Evaluate

\[
\int_C y \, dx + x^2 \, dy
\]

where \( C \) is the parabolic arc given by \( y = 4x - x^2 \) from \((4, 0)\) to \((1, 3)\), as shown in Figure 15.18.

**Solution** Rather than converting to the parameter \( t \), you can simply retain the variable \( x \) and write

\[
y = 4x - x^2 \quad \Rightarrow \quad dy = (4 - 2x) \, dx.
\]

Then, in the direction from \((4, 0)\) to \((1, 3)\), the line integral is

\[
\int_C y \, dx + x^2 \, dy = \int_4^1 [(4x - x^2) \, dx + x^2(4 - 2x) \, dx] \\
= \int_4^1 (4x + 3x^2 - 2x^3) \, dx \\
= \left[ 2x^2 + x^3 - \frac{x^4}{2} \right]_4^1 = \frac{69}{2}.
\]

See Example 7.

**Try It**

**Exploration A**

**EXPLORATION**

**Finding Lateral Surface Area** The figure below shows a piece of tin that has been cut from a circular cylinder. The base of the circular cylinder is modeled by \( x^2 + y^2 = 9 \). At any point \((x, y)\) on the base, the height of the object is given by

\[
f(x, y) = 1 + \cos \frac{\pi x}{4}.
\]

Explain how to use a line integral to find the surface area of the piece of tin.
Exercises for Section 15.2

In Exercises 1–6, find a piecewise smooth parametrization of the path \( C \).

1. \( x^2 + y^2 = 9 \)

2. \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \)

3. \( y = \sqrt{x} \) \( (1, 1) \)

4. \( y = x^2 \) \( (2, 4) \)

5. \( y = \sqrt{x} \) \( (1, 1) \)

6. \( y = x^2 \) \( (2, 4) \)

In Exercises 7–10, evaluate the line integral along the given path.

7. \( \int_C (x - y) \, ds \)
   \( C: \mathbf{r}(t) = 4\mathbf{i} + 3\mathbf{j} \)
   \( 0 \leq t \leq 2 \)

8. \( \int_C 4xy \, ds \)
   \( C: \mathbf{r}(t) = t\mathbf{i} + (2 - t)\mathbf{j} \)
   \( 0 \leq t \leq 2 \)

9. \( \int_C (x^2 + y^2 + z^2) \, ds \)
   \( C: \mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 8t\mathbf{k} \)
   \( 0 \leq t \leq \frac{\pi}{2} \)

10. \( \int_C 8xyz \, ds \)
    \( C: \mathbf{r}(t) = 12\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \)
    \( 0 \leq t \leq 2 \)

In Exercises 11–14, evaluate

\( \int_C (x^2 + y^2) \, ds \)

along the given path.

11. \( C: \) along the path shown in the figure.
12. \( C: \) along the path shown in the figure.
13. \( C: \) counterclockwise around the circle \( x^2 + y^2 = 1 \) from \((1, 0)\) to \((0, 1)\)
14. \( C: \) counterclockwise around the circle \( x^2 + y^2 = 4 \) from \((2, 0)\) to \((0, 2)\)

In Exercises 15–18, evaluate

\( \int_C (x + 4\sqrt{y}) \, ds \)

along the given path.

15. \( C: \) line from \((0, 0)\) to \((1, 1)\)
16. \( C: \) line from \((0, 0)\) to \((3, 9)\)
17. \( C: \) counterclockwise around the triangle with vertices \((0, 0), (1, 0),\) and \((0, 1)\)
18. \( C: \) counterclockwise around the square with vertices \((0, 0), (2, 0), (2, 2),\) and \((0, 2)\)

In Exercises 19 and 20, evaluate

\( \int_C (2x + y^2 - z) \, ds \)

along the path \( C \) shown in the figure.

19. 
20. 

Mass  In Exercises 21 and 22, find the total mass of two turns of a spring with density \( \rho \) in the shape of the circular helix

\( r(t) = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j} + 2t\mathbf{k} \)

21. \( \rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) \)
22. \( \rho(x, y, z) = z \)

Mass  In Exercises 23–26, find the total mass of the wire with density \( \rho \).

23. \( r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} \), \( \rho(x, y) = x + y \), \( 0 \leq t \leq \pi \)
24. \( r(t) = t^2\mathbf{i} + 2t\mathbf{j} \), \( \rho(x, y) = \frac{3}{4}y \), \( 0 \leq t \leq 1 \)
25. \( r(t) = r^2\mathbf{i} + 2t\mathbf{j} + 3k \), \( \rho(x, y, z) = k^2 (k > 0) \), \( 1 \leq t \leq 3 \)
26. \( r(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 3t\mathbf{k} \), \( \rho(x, y, z) = k^2 + z \)
   \( (k > 0) \), \( 0 \leq t \leq 2\pi \)
In Exercises 27–32, evaluate
\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]
where \( C \) is represented by \( \mathbf{r}(t) \).

27. \( \mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j} \)
   \( C: \mathbf{r}(t) = 4t\mathbf{i} + \mathbf{j}, \quad 0 \leq t \leq 1 \)

28. \( \mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j} \)
   \( C: \mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}, \quad 0 \leq t \leq \pi/2 \)

29. \( \mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j} \)
   \( C: \mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad 0 \leq t \leq \pi/2 \)

30. \( \mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j} \)
   \( C: \mathbf{r}(t) = \mathbf{i} + \sqrt{4 - t^2}\mathbf{j}, \quad -2 \leq t \leq 2 \)

31. \( \mathbf{F}(x, y, z) = x^2\mathbf{i} + (x - z)\mathbf{j} + xy\mathbf{k} \)
   \( C: \mathbf{r}(t) = ti + \sqrt{2}t\mathbf{j} + 2t\mathbf{k}, \quad 0 \leq t \leq 1 \)

32. \( \mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k} \)
   \( C: \mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}, \quad 0 \leq t \leq \pi \)

In Exercises 33 and 34, use a computer algebra system to evaluate the integral
\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]
where \( C \) is represented by \( \mathbf{r}(t) \).

33. \( \mathbf{F}(x, y, z) = x^2\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k} \)
   \( C: \mathbf{r}(t) = ti + t^2\mathbf{j} + \ln t\mathbf{k}, \quad 1 \leq t \leq 3 \)

34. \( \mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} \)
   \( C: \mathbf{r}(t) = ti + t\mathbf{j} + e^t\mathbf{k}, \quad 0 \leq t \leq 2 \)

Work In Exercises 35–40, find the work done by the force field \( \mathbf{F} \) on a particle moving along the given path.

35. \( \mathbf{F}(x, y) = -x\mathbf{i} - 2y\mathbf{j} \)
   \( C: y = x^3 \) from \((0, 0)\) to \((2, 8)\)

36. \( \mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j} \)
   \( C: x = \cos^3 t, y = \sin^3 t \) from \((1, 0)\) to \((0, 1)\)

Figure for 36

37. \( \mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j} \)
   \( C: \) counterclockwise around the triangle with vertices \((0, 0)\), \((1, 0)\), and \((1, 1)\)

38. \( \mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j} \)
   \( C: \) counterclockwise along the semicircle \( y = \sqrt{4 - x^2} \) from \((2, 0)\) to \((-2, 0)\)

39. \( \mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k} \)
   \( C: \mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 2\pi \)
40. \( \mathbf{F}(x, y, z) = yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k} \)
    
    \( \mathbf{C} \): line from \((0, 0, 0)\) to \((5, 3, 2)\)

```
\[ \int_C (x + 3y^2) \, dy \]
```

41. Work  
Find the work done by a person weighing 150 pounds walking exactly one revolution up a circular helical staircase of radius 3 feet if the person rises 10 feet.

42. Work  
A particle moves along the path \( y = x^2 \) from the point \((0, 0)\) to the point \((1, 1)\). The force field \( \mathbf{F} \) is measured at five points along the path and the results are shown in the table. Use Simpson’s Rule or a graphing utility to approximate the work done by the force field.

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(0, 0)</th>
<th>((\frac{1}{4}, 1))</th>
<th>((\frac{3}{4}, 2))</th>
<th>((\frac{3}{4}, 0))</th>
<th>((1, 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbf{F}(x, y))</td>
<td>(5, 0)</td>
<td>(3.5, 1)</td>
<td>(2, 2)</td>
<td>(1.5, 3)</td>
<td>(1, 5)</td>
</tr>
</tbody>
</table>

In Exercises 43 and 44, evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) for each curve. Discuss the orientation of the curve and its effect on the value of the integral.

43. \( \mathbf{F}(x, y) = x^2 \mathbf{i} + xy \mathbf{j} \)
   
   (a) \( \mathbf{r}_1(t) = 2t \mathbf{i} + (t - 1) \mathbf{j}, \quad 1 \leq t \leq 3 \)
   
   (b) \( \mathbf{r}_2(t) = 2(3 - t) \mathbf{i} + (2 - t) \mathbf{j}, \quad 0 \leq t \leq 2 \)

44. \( \mathbf{F}(x, y) = x^2y \mathbf{i} + xy^{3/2} \mathbf{j} \)
   
   (a) \( \mathbf{r}_1(t) = (t + 1) \mathbf{i} + r^2 \mathbf{j}, \quad 0 \leq t \leq \frac{1}{2} \)
   
   (b) \( \mathbf{r}_2(t) = (1 + 2 \cos t) \mathbf{i} + (4 \cos^2 t) \mathbf{j}, \quad 0 \leq t \leq \pi/2 \)

In Exercises 45–48, demonstrate the property that

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = 0
\]

regardless of the initial and terminal points of \( \mathbf{C} \), if the tangent vector \( \mathbf{r}'(t) \) is orthogonal to the force field \( \mathbf{F} \).

45. \( \mathbf{F}(x, y) = y \mathbf{i} - x \mathbf{j} \)
    
    \( \mathbf{C}: \mathbf{r}(t) = ti - 2tj \)

46. \( \mathbf{F}(x, y) = -3y \mathbf{i} + x \mathbf{j} \)
    
    \( \mathbf{C}: \mathbf{r}(t) = ti - r^3 \mathbf{j} \)

47. \( \mathbf{F}(x, y) = (x^2 - 2x^2) \mathbf{i} + \left( x - \frac{y^2}{2} \right) \mathbf{j} \)
    
    \( \mathbf{C}: \mathbf{r}(t) = ti + r^2 \mathbf{j} \)

48. \( \mathbf{F}(x, y) = xi + y \mathbf{j} \)
    
    \( \mathbf{C}: \mathbf{r}(t) = 3 \sin t \mathbf{i} + 3 \cos t \mathbf{j} \)

In Exercises 49–52, evaluate the line integral along the path \( \mathbf{C} \) given by \( x = 2t, y = 10t \), where \( 0 \leq t \leq 1 \).

49. \( \int_C (x + 3y^2) \, dy \)

50. \( \int_C (x + 3y^2) \, dx \)

51. \( \int_C xy \, dx + y \, dy \)

52. \( \int_C (3y - x) \, dx + y^2 \, dy \)

In Exercises 53–60, evaluate the integral

\[
\int_C (2x - y) \, dx + (x + 3y) \, dy
\]

along the path \( \mathbf{C} \).

53. \( \mathbf{C}: \) \( x \)-axis from \( x = 0 \) to \( x = 5 \)

54. \( \mathbf{C}: \) \( y \)-axis from \( y = 0 \) to \( y = 2 \)

55. \( \mathbf{C}: \) line segments from \((0, 0)\) to \((3, 0)\) and \((3, 0)\) to \((3, 3)\)

56. \( \mathbf{C}: \) line segments from \((0, 0)\) to \((0, -3)\) and \((0, -3)\) to \((2, -3)\)

57. \( \mathbf{C}: \) arc on \( y = 1 - x^2 \) from \((0, 1)\) to \((1, 0)\)

58. \( \mathbf{C}: \) arc on \( y = x^{3/2} \) from \((0, 0)\) to \((4, 8)\)

59. \( \mathbf{C}: \) parabolic path \( x = t, y = 2t^2 \) from \((0, 0)\) to \((2, 8)\)

60. \( \mathbf{C}: \) elliptic path \( x = 4 \sin t, y = 3 \cos t \) from \((0, 0)\) to \((4, 0)\)

**Lateral Surface Area**  
In Exercises 61–68, find the area of the lateral surface (see figure) over the curve \( \mathbf{C} \) in the \( xy \)-plane and under the surface \( z = f(x, y) \), where

**Lateral surface area** \( = \int_C f(x, y) \, ds \).
69. **Engine Design** A tractor engine has a steel component with a circular base modeled by the vector-valued function \( \mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} \). Its height is given by \( z = 1 + y^2 \). (All measurements of the component are given in centimeters.)
   
   (a) Find the lateral surface area of the component.
   
   (b) The component is in the form of a shell of thickness 0.2 centimeter. Use the result of part (a) to approximate the amount of steel used in its manufacture.
   
   (c) Draw a sketch of the component.

70. **Building Design** The ceiling of a building has a height above the floor given by \( z = 20 + \frac{1}{2} x \), and one of the walls follows a path modeled by \( y = x^{3/2} \). Find the surface area of the wall if \( 0 \leq x \leq 40 \). (All measurements are given in feet.)

**Moments of Inertia** Consider a wire of density \( \rho(x, y) \) given by the space curve

\[
\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j}, \quad a \leq t \leq b.
\]

The moments of inertia about the \( x \)- and \( y \)-axes are given by

\[
I_x = \int_C x^2 \rho(x, y) \, ds
\]

\[
I_y = \int_C y^2 \rho(x, y) \, ds.
\]

In Exercises 71 and 72, find the moments of inertia for the wire of density \( \rho \).

71. A wire lies along \( \mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi \) and \( a > 0 \), with density \( \rho(x, y) = 1 \). 

72. A wire lies along \( \mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi \) and \( a > 0 \), with density \( \rho(x, y) = y \).

**Approximation** In Exercises 73 and 74, determine which value best approximates the lateral surface area over the curve \( C \) in the \( xy \)-plane and under the surface \( z = f(x, y) \). (Make your selection on the basis of a sketch of the surface and not by performing any calculations.)

73. \( f(x, y) = e^{xy} \)

   C: line from \((0, 0)\) to \((2, 2)\)

   (a) 54  (b) 25  (c) \(-250\)  (d) 75  (e) 100

74. \( f(x, y) = y \)

   C: \( y = x^2 \) from \((0, 0)\) to \((2, 4)\)

   (a) 2  (b) 4  (c) 8  (d) 16

75. **Investigation** The top outer edge of a solid with vertical sides and resting on the \( xy \)-plane is modeled by \( \mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + (1 + \sin^2 2t) \mathbf{k} \), where all measurements are in centimeters. The intersection of the plane \( y = b (-3 < b < 3) \) with the top of the solid is a horizontal line.

   (a) Use a computer algebra system to graph the solid.
   
   (b) Use a computer algebra system to approximate the lateral surface area of the solid.
   
   (c) Find (if possible) the volume of the solid.

76. **Investigation** Determine the value of \( c \) such that the work done by the force field

\[
\mathbf{F}(x, y) = 15(4 - x^2)\mathbf{i} - xy\mathbf{j}
\]

on an object moving along the parabolic path \( y = c(1 - x^2) \), between the points \((-1, 0)\) and \((1, 0)\) is a minimum. Compare the result with the work required to move the object along the straight-line path connecting the points.

**Writing About Concepts**

77. Define a line integral of a function \( f \) along a smooth curve \( C \) in the plane and in space. How do you evaluate the line integral as a definite integral?

78. Define a line integral of a continuous vector field \( \mathbf{F} \) on a smooth curve \( C \). How do you evaluate the line integral as a definite integral?

79. Order the surfaces in ascending order of the lateral surface area through the force field shown in the figure is positive, negative, or zero. Explain your answer.

80. For each of the following, determine whether the work done in moving an object from the first to the second point through the force field shown in the figure is positive, negative, or zero. Explain your answer.

   (a) From \((-3, -3)\) to \((3, 3)\)
   
   (b) From \((-3, 0)\) to \((0, 3)\)
   
   (c) From \((5, 0)\) to \((0, 3)\)

**True or False?** In Exercises 81–84, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

81. If \( C \) is given by \( x(t) = t, \; y(t) = t, \; 0 \leq t \leq 1 \), then

\[
\int_C xy \, ds = \int_0^1 t^2 \, dt.
\]

82. If \( C_2 = -C_1 \), then \( \int_{C_2} f(x, y) \, ds = \int_{C_1} f(x, y) \, ds = 0 \).

83. The vector functions \( \mathbf{r}_1 = t \mathbf{i} + t^2 \mathbf{j}, \; 0 \leq t \leq 1 \), and \( \mathbf{r}_2 = (1 - t) \mathbf{i} + (1 - t)^2 \mathbf{j}, \; 0 \leq t \leq 1 \), define the same curve.

84. If \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds = 0 \), then \( \mathbf{F} \) and \( \mathbf{T} \) are orthogonal.

85. **Work** Consider a particle that moves through the force field \( \mathbf{F}(x, y) = (y - x)\mathbf{i} + xy\mathbf{j} \) from the point \((0, 0)\) to the point \((0, 1)\) along the curve \( x = kt(1 - t), \; y = t \). Find the value of \( k \) such that the work done by the force field is 1.
Fundamental Theorem of Line Integrals

The discussion in the preceding section pointed out that in a gravitational field the work done by gravity on an object moving between two points in the field is independent of the path taken by the object. In this section, you will study an important generalization of this result—it is called the **Fundamental Theorem of Line Integrals**.

To begin, an example is presented in which the line integral of a conservative vector field is evaluated over three different paths.

**EXAMPLE 1 Line Integral of a Conservative Vector Field**

Find the work done by the force field

\[ \mathbf{F}(x, y) = \frac{1}{2}xy \mathbf{i} + \frac{1}{4}x^2 \mathbf{j} \]

on a particle that moves from (0, 0) to (1, 1) along each path, as shown in Figure 15.19.  

(a) \( C_1: y = x \)

(b) \( C_2: x = y^2 \)

(c) \( C_3: y = x^3 \)

**Solution**

a. Let \( \mathbf{r}(t) = t \mathbf{i} + t \mathbf{j} \) for \( 0 \leq t \leq 1 \), so that

\[ d\mathbf{r} = (\mathbf{i} + \mathbf{j}) \, dt \quad \text{and} \quad \mathbf{F}(x, y) = \frac{1}{2}t^2 \mathbf{i} + \frac{1}{4}t^2 \mathbf{j}. \]

Then, the work done is

\[ W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left( \frac{1}{2} t^3 \right) \, dt = \left[ \frac{1}{4} t^4 \right]_0^1 = \frac{1}{4}. \]

b. Let \( \mathbf{r}(t) = t \mathbf{i} + \sqrt{t} \mathbf{j} \) for \( 0 \leq t \leq 1 \), so that

\[ d\mathbf{r} = \left( \mathbf{i} + \frac{1}{2\sqrt{t}} \mathbf{j} \right) \, dt \quad \text{and} \quad \mathbf{F}(x, y) = \frac{1}{2}t^{3/2} \mathbf{i} + \frac{1}{4}t^2 \mathbf{j}. \]

Then, the work done is

\[ W = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left( \frac{5}{8} t^{3/2} \right) \, dt = \left[ \frac{1}{4} t^{5/2} \right]_0^1 = \frac{1}{4}. \]

c. Let \( \mathbf{r}(t) = \frac{1}{2} t \mathbf{i} + \frac{3}{8} t^2 \mathbf{j} \) for \( 0 \leq t \leq 2 \), so that

\[ d\mathbf{r} = \left( \frac{1}{2} \mathbf{i} + \frac{3}{8} t \mathbf{j} \right) \, dt \quad \text{and} \quad \mathbf{F}(x, y) = \frac{1}{32} t^4 \mathbf{i} + \frac{1}{16} t^2 \mathbf{j}. \]

Then, the work done is

\[ W = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \frac{5}{128} t^4 \, dt = \left[ \frac{1}{128} t^5 \right]_0^1 = \frac{1}{4}. \]

So, the work done by a conservative vector field is the same for all paths.
NOTE Notice how the Fundamental Theorem of Line Integrals is similar to the Fundamental Theorem of Calculus (Section 4.4), which states that

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]

where \( F'(x) = f(x) \).

In Example 1, note that the vector field \( \mathbf{F}(x, y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{2}x^2\mathbf{j} \) is conservative because \( \mathbf{F}(x, y) = \nabla f(x, y) \), where \( f(x, y) = \frac{1}{4}x^2y \). In such cases, the following theorem states that the value of \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is given by

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(x(1), y(1)) - f(x(0), y(0))
\]

\[
= \frac{1}{4} - 0
\]

\[
= \frac{1}{4}.
\]

**Theorem 15.5** Fundamental Theorem of Line Integrals

Let \( C \) be a piecewise smooth curve lying in an open region \( R \) and given by

\[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \leq t \leq b. \]

If \( \mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j} \) is conservative in \( R \), and \( M \) and \( N \) are continuous in \( R \), then

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))
\]

where \( f \) is a potential function of \( \mathbf{F} \). That is, \( \mathbf{F}(x, y) = \nabla f(x, y) \).

**Proof** A proof is provided only for a smooth curve. For piecewise smooth curves, the procedure is carried out separately on each smooth portion. Because \( \mathbf{F}(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}(x, y)\mathbf{j} \), it follows that

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \, dt
\]

\[
= \int_a^b \left[ \frac{\partial f}{\partial x}(x(t), y(t)) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \frac{dy}{dt} \right] dt
\]

and, by the Chain Rule (Theorem 13.6), you have

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \frac{d}{dt}f(x(t), y(t)) \, dt
\]

\[
= f(x(b), y(b)) - f(x(a), y(a)).
\]

The last step is an application of the Fundamental Theorem of Calculus.

In space, the Fundamental Theorem of Line Integrals takes the following form. Let \( C \) be a piecewise smooth curve lying in an open region \( Q \) and given by

\[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b. \]

If \( \mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \) is conservative and \( M, N, \) and \( P \) are continuous, then

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}
\]

\[
= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))
\]

where \( \mathbf{F}(x, y, z) = \nabla f(x, y, z) \).

The Fundamental Theorem of Line Integrals states that if the vector field \( \mathbf{F} \) is conservative, then the line integral between any two points is simply the difference in the values of the potential function \( f \) at these points.
\textbf{EXAMPLE 2 Using the Fundamental Theorem of Line Integrals}

Evaluate $\int_C \mathbf{F} \cdot \, d\mathbf{r}$, where $C$ is a piecewise smooth curve from $(-1, 4)$ to $(1, 2)$ and \[ \mathbf{F}(x, y) = 2xy \mathbf{i} + (x^2 - y) \mathbf{j} \]
as shown in Figure 15.20.

\textbf{Solution} From Example 6 in Section 15.1, you know that $\mathbf{F}$ is the gradient of $f$ where \[ f(x, y) = x^2y - \frac{y^2}{2} + K. \]

Consequently, $\mathbf{F}$ is conservative, and by the Fundamental Theorem of Line Integrals, it follows that \[
\int_C \mathbf{F} \cdot \, d\mathbf{r} = f(1, 2) - f(-1, 4) \\
= \left[ 1^2(2) - \frac{2^2}{2} \right] - \left[ (-1)^2(4) - \frac{4^2}{2} \right] \\
= 4.
\]

Note that it is unnecessary to include a constant $K$ as part of $f$, because it is canceled by subtraction.

\textbf{Try It}  \textbf{Exploration A}

\textbf{EXAMPLE 3 Using the Fundamental Theorem of Line Integrals}

Evaluate $\int_C \mathbf{F} \cdot \, d\mathbf{r}$, where $C$ is a piecewise smooth curve from $(1, 1, 0)$ to $(0, 2, 3)$ and \[ \mathbf{F}(x, y, z) = 2xy \mathbf{i} + (x^2 + z^2) \mathbf{j} + 2yz \mathbf{k} \]
as shown in Figure 15.21.

\textbf{Solution} From Example 8 in Section 15.1, you know that $\mathbf{F}$ is the gradient of $f$ where \[ f(x, y, z) = x^2y + yz^2 + K. \] Consequently, $\mathbf{F}$ is conservative, and by the Fundamental Theorem of Line Integrals, it follows that \[
\int_C \mathbf{F} \cdot \, d\mathbf{r} = f(0, 2, 3) - f(1, 1, 0) \\
= \left[ (0)^2(2) + (2)^2(3) \right] - \left[ (1)^2(1) + (1)^2(0) \right] \\
= 17.
\]

\textbf{Try It}  \textbf{Exploration A}

In Examples 2 and 3, be sure you see that the value of the line integral is the same for any smooth curve $C$ that has the given initial and terminal points. For instance, in Example 3, try evaluating the line integral for the curve given by \[ \mathbf{r}(t) = (1 - t) \mathbf{i} + (1 + t) \mathbf{j} + 3t \mathbf{k}. \]

You should obtain \[
\int_C \mathbf{F} \cdot \, d\mathbf{r} = \int_0^1 (30t^2 + 16t - 1) \, dt \\
= 17.
\]
Independence of Path

From the Fundamental Theorem of Line Integrals it is clear that if $\mathbf{F}$ is continuous and conservative in an open region $R$, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same for every piecewise smooth curve $C$ from one fixed point in $R$ to another fixed point in $R$. This result is described by saying that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in the region $R$.

A region in the plane (or in space) is connected if any two points in the region can be joined by a piecewise smooth curve lying entirely within the region, as shown in Figure 15.22. In open regions that are connected, the path independence of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is equivalent to the condition that $\mathbf{F}$ is conservative.

**THEOREM 15.6** Independence of Path and Conservative Vector Fields

If $\mathbf{F}$ is continuous on an open connected region, then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path if and only if $\mathbf{F}$ is conservative.

**Proof** If $\mathbf{F}$ is conservative, then, by the Fundamental Theorem of Line Integrals, the line integral is independent of path. Now establish the converse for a plane region $R$. Let $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, and let $(x_0, y_0)$ be a fixed point in $R$. If $(x, y)$ is any point in $R$, choose a piecewise smooth curve $C$ running from $(x_0, y_0)$ to $(x, y)$, and define $f$ by

$$f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dy.$$

The existence of $C$ in $R$ is guaranteed by the fact that $R$ is connected. You can show that $f$ is a potential function of $\mathbf{F}$ by considering two different paths between $(x_0, y_0)$ and $(x, y)$. For the first path, choose $(x_1, y)$ in $R$ such that $x \neq x_1$. This is possible because $R$ is open. Then choose $C_1$ and $C_2$, as shown in Figure 15.23. Using the independence of path, it follows that

$$f(x, y) = \int_{C_1} M\, dx + N\, dy = \int_{C_1} M\, dx + N\, dy + \int_{C_2} M\, dx + N\, dy.$$

Because the first integral does not depend on $x$, and because $dy = 0$ in the second integral, you have

$$f(x, y) = g(y) + \int_{C_2} M\, dx$$

and it follows that the partial derivative of $f$ with respect to $x$ is $f_x(x, y) = M$. For the second path, choose a point $(x, y_1)$. Using reasoning similar to that used for the first path, you can conclude that $f_y(x, y) = N$. Therefore,

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = M\mathbf{i} + N\mathbf{j} = \mathbf{F}(x, y)$$

and it follows that $\mathbf{F}$ is conservative.
**Example 4** Finding Work in a Conservative Force Field

For the force field given by
\[ \mathbf{F}(x, y, z) = e^x \cos y \mathbf{i} - e^x \sin y \mathbf{j} + 2 \mathbf{k} \]
show that \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path, and calculate the work done by \( \mathbf{F} \) on an object moving along a curve \( C \) from \((0, \pi/2, 1)\) to \((1, \pi, 3)\).

**Solution** Writing the force field in the form \( \mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \), you have
\[
\begin{align*}
\frac{\partial P}{\partial y} &= 0 = \frac{\partial N}{\partial z} \\
\frac{\partial P}{\partial x} &= 0 = \frac{\partial M}{\partial z} \\
\frac{\partial N}{\partial x} &= -e^x \sin y = \frac{\partial M}{\partial y}.
\end{align*}
\]

So, \( \mathbf{F} \) is conservative. If \( f \) is a potential function of \( \mathbf{F} \), then
\[
\begin{align*}
f_x(x, y, z) &= e^x \cos y \\
f_y(x, y, z) &= -e^x \sin y \\
f_z(x, y, z) &= 2.
\end{align*}
\]

By integrating with respect to \( x, y, \) and \( z \) separately, you obtain
\[
\begin{align*}
f(x, y, z) &= \int f_x(x, y, z) \, dx = \int e^x \cos y \, dx = e^y \cos y + g(y, z) \\
f(x, y, z) &= \int f_y(x, y, z) \, dy = \int -e^x \sin y \, dy = e^x \cos y + h(x, z) \\
f(x, y, z) &= \int f_z(x, y, z) \, dz = \int 2 \, dz = 2z + k(x, y).
\end{align*}
\]

By comparing these three versions of \( f(x, y, z) \), you can conclude that
\[ f(x, y, z) = e^x \cos y + 2z + K. \]

Therefore, the work done by \( \mathbf{F} \) along any curve \( C \) from \((0, \pi/2, 1)\) to \((1, \pi, 3)\) is
\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \left[ e^x \cos y + 2z \right]^{(1, \pi, 3)}_{(0, \pi/2, 1)} = (-e + 6) - (0 + 2) = 4 - e.
\]

**Try It**

How much work would be done if the object in Example 4 moved from the point \((0, \pi/2, 1)\) to \((1, \pi, 3)\) and then back to the starting point \((0, \pi/2, 1)\)? The Fundamental Theorem of Line Integrals states that there is zero work done. Remember that, by definition, work can be negative. So, by the time the object gets back to its starting point, the amount of work that registers positively is canceled out by the amount of work that registers negatively.
A curve \( C \) given by \( \mathbf{r}(t) \) for \( a \leq t \leq b \) is **closed** if \( \mathbf{r}(a) = \mathbf{r}(b) \). By the Fundamental Theorem of Line Integrals, you can conclude that if \( \mathbf{F} \) is continuous and conservative on an open region \( R \), then the line integral over every closed curve \( C \) is 0.

### Theorem 15.7 Equivalent Conditions

Let \( \mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + P \mathbf{k} \) have continuous first partial derivatives in an open connected region \( R \), and let \( C \) be a piecewise smooth curve in \( R \). The following conditions are equivalent.

1. \( \mathbf{F} \) is conservative. That is, \( \mathbf{F} = \nabla f \) for some function \( f \).
2. \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path.
3. \( \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \) for every closed curve \( C \) in \( R \).

### Example 5 Evaluating a Line Integral

Evaluate \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \), where

\[
\mathbf{F}(x, y) = (y^3 + 1) \mathbf{i} + (3xy^2 + 1) \mathbf{j}
\]

and \( C_1 \) is the semicircular path from \((0, 0)\) to \((2, 0)\), as shown in Figure 15.24.

**Solution** You have the following three options.

a. You can use the method presented in the preceding section to evaluate the line integral along the given curve. To do this, you can use the parametrization \( \mathbf{r}(t) = (1 - \cos t) \mathbf{i} + \sin t \mathbf{j} \), where \( 0 \leq t \leq \pi \). For this parametrization, it follows that \( d\mathbf{r} = \mathbf{r}'(t) \ dt = (\sin t \mathbf{i} + \cos t \mathbf{j}) \ dt \), and

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (\sin t + \sin^3 t + \cos t + 3 \sin^2 t \cos t - 3 \sin^2 t \cos^2 t) \ dt.
\]

This integral should dampen your enthusiasm for this option.

b. You can try to find a **potential function** and evaluate the line integral by the Fundamental Theorem of Line Integrals. Using the technique demonstrated in Example 4, you can find the potential function to be \( f(x, y) = xy^3 + x + y + K \), and, by the Fundamental Theorem,

\[
W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(2, 0) - f(0, 0) = 2.
\]

c. Knowing that \( \mathbf{F} \) is conservative, you have a third option. Because the value of the line integral is independent of path, you can replace the semicircular path with a **simpler path**. Suppose you choose the straight-line path \( C_2 \) from \((0, 0)\) to \((2, 0)\). Then, \( \mathbf{r}(t) = t \mathbf{i} \), where \( 0 \leq t \leq 2 \). So, \( d\mathbf{r} = t \ dt \) and \( \mathbf{F}(x, y) = (y^3 + 1) \mathbf{i} + (3xy^2 + 1) \mathbf{j} = \mathbf{i} + \mathbf{j} \), so that

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 t \ dt = t^2|_0^2 = 2.
\]

Of the three options, obviously the third one is the easiest.
**Conservation of Energy**

In 1840, the English physicist Michael Faraday wrote, “Nowhere is there a pure creation or production of power without a corresponding exhaustion of something to supply it.” This statement represents the first formulation of one of the most important laws of physics—the Law of Conservation of Energy. In modern terminology, the law is stated as follows: In a conservative force field, the sum of the potential and kinetic energies of an object remains constant from point to point.

You can use the Fundamental Theorem of Line Integrals to derive this law. From physics, the kinetic energy of a particle of mass \( m \) and speed \( v \) is \( k = \frac{1}{2}mv^2 \). The potential energy \( p \) of a particle at point \((x, y, z)\) in a conservative vector field \( \mathbf{F} \) is defined as \( p(x, y, z) = -f(x, y, z) \), where \( f \) is the potential function for \( \mathbf{F} \). Consequently, the work done by \( \mathbf{F} \) along a smooth curve \( C \) from \( A \) to \( B \) is

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(x, y, z) \bigg|_A^B = -p(x, y, z) \bigg|_A^B = p(A) - p(B)
\]

as shown in Figure 15.25. In other words, work \( W \) is equal to the difference in the potential energies of \( A \) and \( B \). Now, suppose that \( \mathbf{r}(t) \) is the position vector for a particle moving along \( C \) from \( A = \mathbf{r}(a) \) to \( B = \mathbf{r}(b) \). At any time \( t \), the particle’s velocity, acceleration, and speed are \( \mathbf{v}(t) = \mathbf{r}'(t) \), \( \mathbf{a}(t) = \mathbf{r}''(t) \), and \( v(t) = \|\mathbf{v}(t)\| \), respectively. So, by Newton’s Second Law of Motion, \( \mathbf{F} = \mathbf{m}a(t) = \mathbf{m}(\mathbf{v}'(t)) \), and the work done by \( \mathbf{F} \) is

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_a^b [\mathbf{m}\mathbf{v}'(t)] \cdot \mathbf{v}(t) \, dt = \int_a^b \mathbf{m}[\mathbf{v}'(t) \cdot \mathbf{v}(t)] \, dt
\]

\[
= \frac{m}{2} \int_a^b \frac{d}{dt} [\|\mathbf{v}(t)\|^2] \, dt = \frac{m}{2} \left[ \|\mathbf{v}(t)\|^2 \right]_a^b = \frac{m}{2} \left[ (v(b))^2 - (v(a))^2 \right] = k(B) - k(A).
\]

Equating these two results for \( W \) produces

\[ p(A) - p(B) = k(B) - k(A) \]
\[ p(A) + k(A) = p(B) + k(B) \]

which implies that the sum of the potential and kinetic energies remains constant from point to point.
Exercises for Section 15.3

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on to view the complete solution of the exercise.
Click on to print an enlarged copy of the graph.

In Exercises 1–4, show that the value of \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is the same for each parametric representation of \( C \).

1. \( \mathbf{F}(x, y) = x^2 \mathbf{i} + xy \mathbf{j} \)
   (a) \( r_1(t) = ti + t^2 \mathbf{j}, \ 0 \leq t \leq 1 \)
   (b) \( r_2(\theta) = \sin \theta \mathbf{i} + \sin^2 \theta \mathbf{j}, \ 0 \leq \theta \leq \frac{\pi}{2} \)

2. \( \mathbf{F}(x, y) = (x^2 + y^2) \mathbf{i} - x \mathbf{j} \)
   (a) \( r_1(t) = ti + \sqrt{t} \mathbf{j}, \ 0 \leq t \leq 4 \)
   (b) \( r_2(w) = w^2 \mathbf{i} + w \mathbf{j}, \ 0 \leq w \leq 2 \)

3. \( \mathbf{F}(x, y) = yi - x \mathbf{j} \)
   (a) \( r_1(\theta) = \sec \theta \mathbf{i} + \tan \theta \mathbf{j}, \ 0 \leq \theta \leq \frac{\pi}{3} \)
   (b) \( r_2(\theta) = \sqrt{t} + 1 \mathbf{i} + \sqrt{t} \mathbf{j}, \ 0 \leq t \leq 3 \)

4. \( \mathbf{F}(x, y) = yi + x^2 \mathbf{j} \)
   (a) \( r_1(t) = (2 + t)i + (3 - t)j, \ 0 \leq t \leq 3 \)
   (b) \( r_2(w) = (2 + \ln w)i + (3 - \ln w)j, \ 1 \leq w \leq e^3 \)

In Exercises 5–10, determine whether or not the vector field is conservative.

5. \( \mathbf{F}(x, y) = e^t (\sin yi + \cos yj) \)
6. \( \mathbf{F}(x, y) = 15x^2y^2 \mathbf{i} + 10x^3y \mathbf{j} \)
7. \( \mathbf{F}(x, y) = \frac{1}{y^2} (yi + xj) \)
8. \( \mathbf{F}(x, y, z) = y \ln zi - x \ln zj + \frac{xy}{z} \mathbf{k} \)
9. \( \mathbf{F}(x, y, z) = y^2zi + 2xyzj + x^3y \mathbf{k} \)
10. \( \mathbf{F}(x, y, z) = \sin yz \mathbf{i} + xz \cos yzj + xy \sin yz \mathbf{k} \)

In Exercises 11–24, find the value of the line integral

\[ \int_C \mathbf{F} \cdot d\mathbf{r} . \]

(Hint: If \( \mathbf{F} \) is conservative, the integration may be easier on an alternative path.)

11. \( \mathbf{F}(x, y) = 2xy \mathbf{i} + x^2 \mathbf{j} \)
    (a) \( r_1(t) = ti + t^2 \mathbf{j}, \ 0 \leq t \leq 1 \)
    (b) \( r_2(\theta) = ti + t^2 \mathbf{j}, \ 0 \leq t \leq 1 \)

12. \( \mathbf{F}(x, y) = ye^y \mathbf{i} + xe^{xy} \mathbf{j} \)
    (a) \( r_1(t) = ti - (t - 3)j, \ 0 \leq t \leq 3 \)
    (b) The closed path consisting of line segments from \((0, 3)\) to \((0, 0)\), and then from \((0, 0)\) to \((3, 0)\)

13. \( \mathbf{F}(x, y) = yi - xj \)
    (a) \( r_1(t) = ti + r^2 \mathbf{j}, \ 0 \leq r \leq 1 \)
    (b) \( r_2(t) = ti + r^2 \mathbf{j}, \ 0 \leq r \leq 1 \)
    (c) \( r_3(t) = ti + r^2 \mathbf{j}, \ 0 \leq t \leq 1 \)

14. \( \mathbf{F}(x, y) = xy^2 \mathbf{i} + 2x^3y \mathbf{j} \)
    (a) \( r_1(t) = ti + \frac{1}{t} \mathbf{j}, \ 1 \leq t \leq 3 \)
    (b) \( r_2(t) = \left(t + 1\right)i - \frac{1}{3}(t - 3) \mathbf{j}, \ 0 \leq t \leq 2 \)

15. \( \int_C y^2 \mathbf{d}x + 2xy \mathbf{d}y \)
16. \( \int_C (2x - 3y + 1) \mathbf{d}x - (3x + y - 5) \mathbf{d}y \)
17. \( \int_C 2xy \mathbf{d}x + (x^2 + y^2) \mathbf{d}y \)
    (a) \( C: \text{ellipse } \frac{x^2}{25} + \frac{y^2}{16} = 1 \text{ from } (5, 0) \text{ to } (0, 4) \)
    (b) \( C: \text{parabola } y = 4 - x^2 \text{ from } (2, 0) \text{ to } (0, 4) \)
18. \( \int_C (x^2 + y^2) \, dx + 2xy \, dy \)
   (a) \( r(t) = t^4 i + t^3 j, \quad 0 \leq t \leq 2 \)
   (b) \( r_2(t) = 2 \cos t i + 2 \sin tj, \quad 0 \leq t \leq \frac{\pi}{2} \)

19. \( \mathbf{F}(x, y, z) = yz i + xzj + xyk \)
   (a) \( r_1(t) = ti + 2j + tk, \quad 0 \leq t \leq 4 \)
   (b) \( r_2(t) = t^2 i + tj + t^2 k, \quad 0 \leq t \leq 2 \)

20. \( \mathbf{F}(x, y, z) = \mathbf{i} + zj + yk \)
   (a) \( r_1(t) = \cos ti + \sin tj + t^2k, \quad 0 \leq t \leq \pi \)
   (b) \( r_2(t) = (1 - 2t)i + \pi tjk, \quad 0 \leq t \leq 1 \)

21. \( \mathbf{F}(x, y, z) = (2y + x)i + (x^2 - z)j + (2y - 4z)k \)
   (a) \( r_1(t) = ti + t^2j + k, \quad 0 \leq t \leq 1 \)
   (b) \( r_2(t) = ti + (t^2 - 1)^2k, \quad 0 \leq t \leq 1 \)

22. \( \mathbf{F}(x, y, z) = -yi + xj + 3x^2k \)
   (a) \( r_1(t) = \cos ti + \sin tj + tk, \quad 0 \leq t \leq \pi \)
   (b) \( r_2(t) = (1 - 2t)i + \pi tjk, \quad 0 \leq t \leq 1 \)

23. \( \mathbf{F}(x, y, z) = e^3 (yi + xj + xyk) \)
   (a) \( r_1(t) = 4 \cos ti + 4 \sin j + 3k, \quad 0 \leq t \leq \pi \)
   (b) \( r_2(t) = (4 - 8t)i + 3j, \quad 0 \leq t \leq 1 \)

24. \( \mathbf{F}(x, y, z) = y \sin z + x \sin z + xy \cos xk \)
   (a) \( r_1(t) = t^2 i + t^2 j, \quad 0 \leq t \leq 2 \)
   (b) \( r_2(t) = 4ri + 4tj, \quad 0 \leq t \leq 1 \)

In Exercises 25–34, evaluate the line integral using the Fundamental Theorem of Line Integrals. Use a computer algebra system to verify your results.

25. \( \int_C (yi + xj) \cdot dr \)
   C: smooth curve from \((0, 0)\) to \((3, 8)\)

26. \( \int_C [2(x + y)i + 2(x + y)j] \cdot dr \)
   C: smooth curve from \((-2, 2)\) to \((4, 3)\)

27. \( \int_C \cos x \sin y \, dx + \sin x \cos y \, dy \)
   C: smooth curve from \((0, -\pi)\) to \((\frac{3\pi}{2}, \frac{\pi}{2})\)

28. \( \int_C \frac{y \, dx - x \, dy}{x^2 + y^2} \)
   C: smooth curve from \((1, 1)\) to \((2\sqrt{3}, 2)\)

29. \( \int_C e^x \sin y \, dx + e^x \cos y \, dy \)
   C: cycloid \(x = \theta - \sin \theta, y = 1 - \cos \theta\) from \((0, 0)\) to \((2\pi, 0)\)

30. \( \int_C \frac{2x \, dx + 2y \, dy}{x^2 + y^2} \)
   C: circle \((x - 4)^2 + (y - 5)^2 = 9\) clockwise from \((7, 5)\) to \((1, 5)\)

31. \( \int_C (y + 2z) \, dx + (x - 3z) \, dy + (2x - 3y) \, dz \)
   (a) \( C\): line segment from \((0, 0, 0)\) to \((1, 1, 1)\)
   (b) \( C\): line segments from \((0, 0, 0)\) to \((0, 0, 1)\) to \((1, 1, 1)\)
   (c) \( C\): line segments from \((0, 0, 0)\) to \((1, 0, 0)\) to \((1, 1, 0)\) to \((1, 1, 1)\)

32. Repeat Exercise 31 using the integral

33. \( \int_C -\sin x \, dx + z \, dy + y \, dz \)
   C: smooth curve from \((0, 0, 0)\) to \((\frac{\pi}{2}, 3, 4)\)

34. \( \int_C 6x \, dx - 4z \, dy - (4y - 20z) \, dz \)
   C: smooth curve from \((0, 0, 0)\) to \((4, 3, 1)\)

35. \( \mathbf{F}(x, y) = 9x^2y^2i + (6x^2y - 1)j; \quad P(0, 0), Q(5, 9) \)

36. \( \mathbf{F}(x, y) = \frac{2x}{y}i - \frac{x^2}{y^2}j; \quad P(-3, 2), Q(1, 4) \)

37. **Work** A stone weighing 1 pound is attached to the end of a two-foot string and is whirled horizontally with one end held fixed. It makes 1 revolution per second. Find the work done by the force \( \mathbf{F} \) that keeps the stone moving in a circular path \([\text{Hint: Use Force} = (\text{mass})(\text{centripetal acceleration}).]\)

38. **Work** If \( \mathbf{F}(x, y, z) = a_1i + a_2j + a_3k \) is a constant force vector field, show that the work done in moving a particle along any path from \( P \) to \( Q \) is \( W = \mathbf{F} \cdot \mathbf{PQ} \)

39. **Work** To allow a means of escape for workers in a hazardous job 50 meters above ground level, a slide wire is installed. It runs from their position to a point on the ground 50 meters from the base of the installation where they are located. Show that the work done by the gravitational force field for a 150-pound man moving the length of the slide wire is the same for each path.
   (a) \( r(t) = ti + (50 - t)j \)
   (b) \( r(t) = ti + \frac{50 - t}{2}j \)

40. **Work** Can you find a path for the slide wire in Exercise 35 such that the work done by the gravitational force field would differ from the amounts of work done for the two paths given? Explain why or why not.

**Writing About Concepts**

41. State the Fundamental Theorem of Line Integrals.

42. What does it mean that a line integral is independent of path? State the method for determining if a line integral is independent of path.
43. Consider the force field shown in the figure.

\[ \int_C \mathbf{F} \cdot d\mathbf{r} \neq 0. \]

(a) Give a verbal argument that the force field is not conservative because you can identify two paths that require different amounts of work to move an object from \((-4, 0)\) to \((3, 4)\). Identify two paths and state which requires the greater amount of work. To print an enlarged copy of the graph, select the MathGraph button.

(b) Give a verbal argument that the force field is not conservative because you can find a closed curve \(C\) such that

\[ \int_C \mathbf{F} \cdot d\mathbf{r} \neq 0. \]

44. Wind Speed and Direction

The map shows the jet stream wind speed vectors over the United States for March 19, 2004. In planning a flight from Dallas to Atlanta in a small plane at an altitude of 5000 feet, is the amount of fuel required independent of the flight path? Is the vector field conservative? Explain.

In Exercises 45 and 46, consider the force field shown in the figure. Is the force field conservative? Explain why or why not.

45. \[ \mathbf{F} \]

46. \[ \mathbf{F} \]

True or False? In Exercises 47–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

47. If \(C_1, C_2,\) and \(C_3\) have the same initial and terminal points and \(\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2,\) then \(\int_{C_3} \mathbf{F} \cdot d\mathbf{r}_3 = \int_{C_4} \mathbf{F} \cdot d\mathbf{r}_4.\)

48. If \(\mathbf{F} = y \mathbf{i} + x \mathbf{j}\) and \(C\) is given by \(r(t) = (4 \sin t) \mathbf{i} + (3 \cos t) \mathbf{j}\) \(0 \leq t \leq \pi,\) then \(\int_C \mathbf{F} \cdot d\mathbf{r} = 0.\)

49. If \(\mathbf{F}\) is conservative in a region \(R\) bounded by a simple closed path \(C\) that lies within \(R,\) then \(\int_C \mathbf{F} \cdot d\mathbf{r}\) is independent of path.

50. If \(\mathbf{F} = M \mathbf{i} + N \mathbf{j}\) and \(\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y},\) then \(\mathbf{F}\) is conservative.

51. A function \(f\) is called harmonic if \(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.\) Prove that if \(f\) is harmonic, then

\[ \int_C \left( \frac{\partial f}{\partial y} \, dx - \frac{\partial f}{\partial x} \, dy \right) = 0 \]

where \(C\) is a smooth closed curve in the plane.

52. Kinetic and Potential Energy

The kinetic energy of an object moving through a conservative force field is decreasing at a rate of 10 units per minute. At what rate is the potential energy changing?

53. Let \(\mathbf{F}(x, y) = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j}.\)

(a) Show that

\[ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \]

where

\[ M = \frac{y}{x^2 + y^2}\] and \(N = \frac{-x}{x^2 + y^2}.\)

(b) If \(r(t) = \cos t \mathbf{i} + \sin t \mathbf{j}\) for \(0 \leq t \leq \pi,\) find \(\int_C \mathbf{F} \cdot d\mathbf{r}.\)

(c) If \(r(t) = \cos t \mathbf{i} - \sin t \mathbf{j}\) for \(0 \leq t \leq \pi,\) find \(\int_C \mathbf{F} \cdot d\mathbf{r}.\)

(d) If \(r(t) = \cos t \mathbf{i} + \sin t \mathbf{j}\) for \(0 \leq t \leq 2\pi,\) find \(\int_C \mathbf{F} \cdot d\mathbf{r}.\)

Why doesn’t this contradict Theorem 15.7?

(e) Show that \(\nabla \left( \arctan \frac{x}{y} \right) = \mathbf{F}.\)
**Section 15.4**

**Green’s Theorem**

- Use Green’s Theorem to evaluate a line integral.
- Use alternative forms of Green’s Theorem.

### Green’s Theorem

In this section, you will study Green’s Theorem, named after the English mathematician George Green (1793–1841). This theorem states that the value of a double integral over a simply connected plane region $R$ is determined by the value of a line integral around the boundary of $R$.

A curve $C$ given by $r(t) = x(t)i + y(t)j$, where $a \leq t \leq b$, is simple if it does not cross itself—that is, $r(c) \neq r(d)$ for all $c$ and $d$ in the open interval $(a, b)$. A plane region $R$ is simply connected if its boundary consists of one simple closed curve, as shown in Figure 15.26.

**THEOREM 15.8 Green’s Theorem**

Let $R$ be a simply connected region with a piecewise smooth boundary $C$, oriented counterclockwise (that is, $C$ is traversed once so that the region $R$ always lies to the left). If $M$ and $N$ have continuous partial derivatives in an open region containing $R$, then

$$
\int_C M \, dx + N \, dy = \int_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.
$$

**Proof**  A proof is given only for a region that is both vertically simple and horizontally simple, as shown in Figure 15.27.

$$
\int_C M \, dx = \int_{C_1} M \, dx + \int_{C_2} M \, dx
= \int_a^b M(x, f_1(x)) \, dx + \int_a^b M(x, f_2(x)) \, dx
= \int_a^b \left[ M(x, f_1(x)) - M(x, f_2(x)) \right] \, dx
$$

On the other hand,

$$
\int_R \frac{\partial M}{\partial y} \, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} \, dy \, dx
= \int_a^b M(x, y) \int_{f_1(x)}^{f_2(x)} \, dy \, dx
= \int_a^b \left[ M(x, f_2(x)) - M(x, f_1(x)) \right] \, dx.
$$

Consequently,

$$
\int_C M \, dx = -\int_R \frac{\partial M}{\partial y} \, dA.
$$

Similarly, you can use $g_1(y)$ and $g_2(y)$ to show that $\int_C N \, dy = \int_R \frac{\partial N}{\partial x} \, dA$. By adding the integrals $\int_C M \, dx$ and $\int_C N \, dy$, you obtain the conclusion stated in the theorem.
EXAMPLE 1 Using Green’s Theorem

Use Green’s Theorem to evaluate the line integral
\[ \int_C y^3 \, dx + (x^3 + 3xy^2) \, dy \]
where \( C \) is the path from \((0, 0)\) to \((1, 1)\) along the graph of \( y = x^3 \) and from \((1, 1)\) to \((0, 0)\) along the graph of \( y = x \), as shown in Figure 15.28.

Solution Because \( M = y^3 \) and \( N = x^3 + 3xy^2 \), it follows that
\[ \frac{\partial N}{\partial x} = 3x^2 + 3y^2 \quad \text{and} \quad \frac{\partial M}{\partial y} = 3y^2. \]

Applying Green’s Theorem, you then have
\[
\int_C y^3 \, dx + (x^3 + 3xy^2) \, dy = \int_{R_1} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \\
= \int_0^1 \int_0^x [(3x^2 + 3y^2) - 3y^2] \, dy \, dx \\
= \int_0^1 \int_0^x 3x^2 \, dy \, dx \\
= \int_0^1 3x^2 \left[ y \right]_0^x \, dx \\
= \int_0^1 (3x^3 - 3x^5) \, dx \\
= \left[ \frac{3x^4}{4} - \frac{x^6}{2} \right]_0^1 \\
= 1 - \frac{1}{4} = \frac{3}{4}.
\]

Try It

Green’s Theorem cannot be applied to every line integral. Among other restrictions stated in Theorem 15.8, the curve \( C \) must be simple and closed. When Green’s Theorem does apply, however, it can save time. To see this, try using the techniques described in Section 15.2 to evaluate the line integral in Example 1. To do this, you would need to write the line integral as
\[
\int_C y^3 \, dx + (x^3 + 3xy^2) \, dy = \int_{C_1} y^3 \, dx + (x^3 + 3xy^2) \, dy + \int_{C_2} y^3 \, dx + (x^3 + 3xy^2) \, dy
\]
where \( C_1 \) is the cubic path given by
\[ \mathbf{r}(t) = ti + t^3j \]
from \( t = 0 \) to \( t = 1 \), and \( C_2 \) is the line segment given by
\[ \mathbf{r}(t) = (1 - t)i + (1 - t)j \]
from \( t = 0 \) to \( t = 1 \).
**EXAMPLE 2 Using Green’s Theorem to Calculate Work**

While subject to the force

\[ \mathbf{F}(x, y) = y^3 \mathbf{i} + (x^3 + 3xy^2) \mathbf{j} \]

a particle travels once around the circle of radius 3 shown in Figure 15.29. Use Green’s Theorem to find the work done by \( \mathbf{F} \).

**Solution** From Example 1, you know by Green’s Theorem that

\[ \int_C y^3 \, dx + (x^3 + 3xy^2) \, dy = \iint_R 3x^2 \, dA. \]

In polar coordinates, using \( x = r \cos \theta \) and \( dA = r \, dr \, d\theta \), the work done is

\[
W = \iint_R 3x^2 \, dA = \int_0^{2\pi} \int_0^3 3(r \cos \theta)^2 \, r \, dr \, d\theta \\
= \frac{3}{2} \int_0^{2\pi} \int_0^3 r^3 \cos^2 \theta \, d\theta \\
= \frac{3}{2} \int_0^{2\pi} \frac{r^4}{4} \cos^2 \theta \, d\theta \\
= \frac{3}{8} \int_0^{2\pi} \frac{81}{4} \cos^2 \theta \, d\theta \\
= \frac{243}{8} \int_0^{2\pi} (1 + \cos 2\theta) \, d\theta \\
= \frac{243}{8} \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{2\pi} \\
= \frac{243\pi}{4}.
\]

**Try It**

When evaluating line integrals over closed curves, remember that for conservative vector fields (those for which \( \partial N/\partial x = \partial M/\partial y \)), the value of the line integral is 0. This is easily seen from the statement of Green’s Theorem:

\[ \int_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = 0. \]

**EXAMPLE 3 Green’s Theorem and Conservative Vector Fields**

Evaluate the line integral

\[ \int_C y^3 \, dx + 3xy^2 \, dy \]

where \( C \) is the path shown in Figure 15.30.

**Solution** From this line integral, \( M = y^3 \) and \( N = 3xy^2 \). So, \( \partial N/\partial x = 3y^2 \) and \( \partial M/\partial y = 3y^2 \). This implies that the vector field \( \mathbf{F} = Mi + Nj \) is conservative, and because \( C \) is closed, you can conclude that

\[ \int_C y^3 \, dx + 3xy^2 \, dy = 0. \]
EXAMPLE 4  Using Green’s Theorem for a Piecewise Smooth Curve

Evaluate
\[ \int_C (\arctan x + y^2) \, dx + (e^y - x^2) \, dy \]
where \(C\) is the path enclosing the annular region shown in Figure 15.31.

Solution  In polar coordinates, \(R\) is given by \(1 \leq r \leq 3\) for \(0 \leq \theta \leq \pi\). Moreover,
\[
\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2x - 2y = -2(r \cos \theta + r \sin \theta).
\]
So, by Green’s Theorem,
\[
\int_C (\arctan x + y^2) \, dx + (e^y - x^2) \, dy = \int_R \int (-2(x + y)) \, dA
\]
\[
= \int_0^\pi \int_1^3 -2r(\cos \theta + \sin \theta) \, r \, dr \, d\theta
\]
\[
= \int_0^\pi -2(\cos \theta + \sin \theta) \frac{r^3}{3} \bigg|_1^3 \, d\theta
\]
\[
= \int_0^\pi \left(-\frac{52}{3}\right)(\cos \theta + \sin \theta) \, d\theta
\]
\[
= -\frac{52}{3} \left[ \sin \theta - \cos \theta \right]_0^\pi
\]
\[
= -\frac{104}{3}.
\]

Try It

In Examples 1, 2, and 4, Green’s Theorem was used to evaluate line integrals as double integrals. You can also use the theorem to evaluate double integrals as line integrals. One useful application occurs when \(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1\).

\[
\int_C M \, dx + N \, dy = \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
\]
\[
= \int_R \int 1 \, dA
\]
\[
= \text{area of region } R
\]
Among the many choices for \(M\) and \(N\) satisfying the stated condition, the choice of \(M = -y/2\) and \(N = x/2\) produces the following line integral for the area of region \(R\).

THEOREM 15.9  Line Integral for Area

If \(R\) is a plane region bounded by a piecewise smooth simple closed curve \(C\), oriented counterclockwise, then the area of \(R\) is given by
\[
A = \frac{1}{2} \int_C x \, dy - y \, dx.
\]
EXAMPLE 5  Finding Area by a Line Integral

Use a line integral to find the area of the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]

Solution  Using Figure 15.32, you can induce a counterclockwise orientation to the elliptical path by letting

\[
x = a \cos t \quad \text{and} \quad y = b \sin t, \quad 0 \leq t \leq 2\pi.
\]

So, the area is

\[
A = \frac{1}{2} \int_C x \, dy - y \, dx = \frac{1}{2} \int_0^{2\pi} [(a \cos t)(b \cos t) \, dt - (b \sin t)(-a \sin t) \, dt]
\]

\[
= \frac{ab}{2} \int_0^{2\pi} (\cos^2 t + \sin^2 t) \, dt
\]

\[
= \frac{ab}{2} \left[ t \right]_0^{2\pi}
\]

\[
= \pi ab.
\]

Try It  Exploration A

Green’s Theorem can be extended to cover some regions that are not simply connected. This is demonstrated in the next example.

EXAMPLE 6  Green’s Theorem Extended to a Region with a Hole

Let \( R \) be the region inside the ellipse \((x^2/9) + (y^2/4) = 1\) and outside the circle \(x^2 + y^2 = 1\). Evaluate the line integral

\[
\int_C 2xy \, dx + (x^2 + 2x) \, dy
\]

where \( C = C_1 + C_2 \) is the boundary of \( R \), as shown in Figure 15.33.

Solution  To begin, you can introduce the line segments \( C_3 \) and \( C_4 \), as shown in Figure 15.33. Note that because the curves \( C_3 \) and \( C_4 \) have opposite orientations, the line integrals over them cancel. Furthermore, you can apply Green’s Theorem to the region \( R \) using the boundary \( C_1 + C_4 + C_2 + C_3 \) to obtain

\[
\int_C 2xy \, dx + (x^2 + 2x) \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
\]

\[
= \iint_R (2x + 2 - 2x) \, dA
\]

\[
= 2 \iint_R dA
\]

\[
= 2(\text{area of } R)
\]

\[
= 2(\pi ab - \pi r^2)
\]

\[
= 2[\pi(3)(2) - \pi(1^2)]
\]

\[
= 10\pi.
\]

Try It  Exploration A
In Section 15.1, a necessary and sufficient condition for conservative vector fields was listed. There, only one direction of the proof was shown. You can now outline the other direction, using Green’s Theorem. Let \( \mathbf{F}(x, y) = M \mathbf{i} + N \mathbf{j} \) be defined on an open disk \( R \). You want to show that if \( M \) and \( N \) have continuous first partial derivatives and

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]

then \( \mathbf{F} \) is conservative. Suppose that \( C \) is a closed path forming the boundary of a connected region lying in \( R \). Then, using the fact that \( \partial M/\partial y = \partial N/\partial x \), you can apply Green’s Theorem to conclude that

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \, dx + N \, dy
\]

\[
= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
\]

\[
= 0.
\]

This, in turn, is equivalent to showing that \( \mathbf{F} \) is conservative (see Theorem 15.7).

**Alternative Forms of Green’s Theorem**

This section concludes with the derivation of two vector forms of Green’s Theorem for regions in the plane. The extension of these vector forms to three dimensions is the basis for the discussion in the remaining sections of this chapter. If \( \mathbf{F} \) is a vector field in the plane, you can write

\[
\mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + 0 \mathbf{k}
\]

so that the curl of \( \mathbf{F} \), as described in Section 15.1, is given by

\[
\text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
M & N & 0
\end{vmatrix}
\]

\[
= -\frac{\partial N}{\partial z} \mathbf{i} + \frac{\partial M}{\partial z} \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}.
\]

Consequently,

\[
(\text{curl} \mathbf{F}) \cdot \mathbf{k} = \left[ -\frac{\partial N}{\partial z} \mathbf{i} + \frac{\partial M}{\partial z} \mathbf{j} + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \right] \cdot \mathbf{k}
\]

\[
= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.
\]

With appropriate conditions on \( \mathbf{F}, C, \) and \( R \), you can write Green’s Theorem in the vector form

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
\]

\[
= \iint_R (\text{curl} \mathbf{F}) \cdot \mathbf{k} \, dA. \quad \text{First alternative form}
\]

The extension of this vector form of Green’s Theorem to surfaces in space produces Stokes’s Theorem, discussed in Section 15.8.
For the second vector form of Green’s Theorem, assume the same conditions for $\mathbf{F}$, $C$, and $R$. Using the arc length parameter $s$ for $C$, you have $\mathbf{r}(s) = x(s) \mathbf{i} + y(s) \mathbf{j}$. So, a unit tangent vector $\mathbf{T}$ to curve $C$ is given by

$$\mathbf{r}'(s) = \mathbf{T} = x'(s) \mathbf{i} + y'(s) \mathbf{j}.$$  

From Figure 15.34 you can see that the outward unit normal vector $\mathbf{N}$ can then be written as

$$\mathbf{N} = y'(s) \mathbf{i} - x'(s) \mathbf{j}.$$  

Consequently, for $\mathbf{F}(x, y) = Mi + Nj$, you can apply Green’s Theorem to obtain

$$\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \int_a^b (M \frac{dy}{ds} - N \frac{dx}{ds}) \, ds$$

$$= \int_a^b M \, dy - N \, dx$$

$$= \int_C -N \, dx + M \, dy$$

$$= \int_{R} \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dA \quad \text{Green’s Theorem}$$

$$= \int_{R} \text{div} \, \mathbf{F} \, dA.$$  

Therefore,

$$\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \int_{R} \text{div} \, \mathbf{F} \, dA. \quad \text{Second alternative form}$$

The extension of this form to three dimensions is called the Divergence Theorem, discussed in Section 15.7. The physical interpretations of divergence and curl will be discussed in Sections 15.7 and 15.8.
Exercises for Section 15.4

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on to view the complete solution of the exercise.
Click on to print an enlarged copy of the graph.

In Exercises 1–4, verify Green’s Theorem by evaluating both integrals

\[ \int_C y^2 \, dx + x^2 \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \]

for the given path.

1. \( C \): square with vertices \((0, 0), (4, 0), (4, 4), (0, 4)\)
2. \( C \): triangle with vertices \((0, 0), (4, 0), (4, 4)\)
3. \( C \): boundary of the region lying between the graphs of \( y = x \) and \( y = x^2/4 \)
4. \( C \): circle given by \( x^2 + y^2 = 1 \)

In Exercises 5 and 6, verify Green’s Theorem by using a computer algebra system to evaluate both integrals

\[ \int_C xe^y \, dx + e^x \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \]

for the given path.

5. \( C \): circle given by \( x^2 + y^2 = 4 \)
6. \( C \): boundary of the region lying between the graphs of \( y = \) and \( y = x^3 \) in the first quadrant

In Exercises 7–10, use Green’s Theorem to evaluate the integral

\[ \int_C (y - x) \, dx + (2x - y) \, dy \]

for the given path.

7. \( C \): boundary of the region lying between the graphs of \( y = x \) and \( y = x^2 - x \)
8. \( C \): \( x = 2 \cos \theta, y = \sin \theta \)
9. \( C \): boundary of the region lying inside the rectangle bounded by \( x = -5, x = 5, y = -3, \) and \( y = 3 \), and outside the square bounded by \( x = -1, x = 1, y = -1, \) and \( y = 1 \)
10. \( C \): boundary of the region lying inside the semicircle \( y = \sqrt{25 - x^2} \) and outside the semicircle \( y = \sqrt{9 - x^2} \)
In Exercises 11–20, use Green’s Theorem to evaluate the line integral.

11. \[ \int_C (2xy + (x + y)) \, dy \]
   \[ C: \text{boundary of the region lying between the graphs of } y = 0 \quad \text{and } y = 4 - x^2 \]

12. \[ \int_C (y^2 + xy) \, dy \]
   \[ C: \text{boundary of the region lying between the graphs of } y = 0, \quad y = \sqrt{x}, \text{and } x = 9 \]

13. \[ \int_C (x^2 - y^2) \, dx + 2xy \, dy \]
   \[ C: x^2 + y^2 = a^2 \]

14. \[ \int_C (x^2 - y^2) \, dx + 2xy \, dy \]
   \[ C: r = 1 + \cos \theta \]

15. \[ \int_C 2 \arctan \frac{y}{x} \, dx + \ln(x^2 + y^2) \, dy \]
   \[ C: x = 4 + 2 \cos \theta, \quad y = 4 + \sin \theta \]

16. \[ \int_C e^x \cos 2y \, dx - 2e^x \sin 2y \, dy \]
   \[ C: x^2 + y^2 = a^2 \]

17. \[ \int_C \sin x \cos y \, dx + (xy + \cos x \sin y) \, dy \]
   \[ C: \text{boundary of the region lying between the graphs of } y = x \quad \text{and } y = \sqrt{x} \]

18. \[ \int_C (e^{-x^{1/2}} - y) \, dx + (e^{-x^{1/2}} + x) \, dy \]
   \[ C: \text{boundary of the region lying between the graphs of the circle } x = 6 \cos \theta, \quad y = 6 \sin \theta \quad \text{and the ellipse } x = 3 \cos \theta, \quad y = 2 \sin \theta \]

19. \[ \int_C xy \, dx + (x + y) \, dy \]
   \[ C: \text{boundary of the region lying between the graphs of } x^2 + y^2 = 1 \quad \text{and } x^2 + y^2 = 9 \]

20. \[ \int_C 3x^2e^y \, dx + e^y \, dy \]
   \[ C: \text{boundary of the region lying between the squares with vertices } (1, 1), \quad (-1, 1), \quad (-1, -1), \quad \text{and} \quad (1, -1), \quad \text{and} \quad (2, 2), \quad (-2, 2), \quad (-2, -2), \quad \text{and} \quad (2, -2) \]

Work In Exercises 21–24, use Green’s Theorem to calculate the work done by the force \( F \) on a particle that is moving counterclockwise around the closed path \( C \).

21. \( F(x, y) = xy \mathbf{i} + (x + y) \mathbf{j} \)
   \[ C: x^2 + y^2 = 4 \]

22. \( F(x, y) = (e^x - 3y) \mathbf{i} + (e^x + 6x) \mathbf{j} \)
   \[ C: r = 2 \cos \theta \]

23. \( F(x, y) = (x^{3/2} - 3y) \mathbf{i} + (6x + 5\sqrt{y}) \mathbf{j} \)
   \[ C: \text{boundary of the triangle with vertices } (0, 0), \quad (5, 0), \quad \text{and} \quad (0, 5) \]

24. \( F(x, y) = (3x^2 + y) \mathbf{i} + 4xy^2 \mathbf{j} \)
   \[ C: \text{boundary of the region lying between the graphs of } y = \sqrt{x}, \quad y = 0, \quad \text{and} \quad x = 9 \]

Area In Exercises 25–28, use a line integral to find the area of the region \( R \).

25. \( R: \text{region bounded by the graph of } x^2 + y^2 = a^2 \)

26. \( R: \text{triangle bounded by the graphs of } x = 0, \quad 3x - 2y = 0, \quad \text{and} \quad x + 2y = 8 \)

27. \( R: \text{region bounded by the graphs of } y = 2x + 1 \quad \text{and} \quad y = 4 - x^2 \)

28. \( R: \text{region inside the loop of the folium of Descartes bounded by the graph of} \)
   \[ x = \frac{3t}{t^3 + 1}, \quad y = \frac{3t^2}{t^3 + 1} \]

Writing About Concepts

29. State Green’s Theorem.

30. Give the line integral for the area of a region \( R \) bounded by a piecewise smooth simple curve \( C \).

In Exercises 31 and 32, use Green’s Theorem to verify the line integral formulas.

31. The centroid of the region having area \( A \) bounded by the simple closed path \( C \) is
   \[ \bar{x} = \frac{1}{2A} \int_C x^2 \, dy, \quad \bar{y} = -\frac{1}{2A} \int_C y^2 \, dx. \]

32. The area of a plane region bounded by the simple closed path \( C \) given in polar coordinates is \( A = \frac{1}{2} \int_C r^2 \, d\theta \).

Centroid In Exercises 33–36, use a computer algebra system and the results of Exercise 31 to find the centroid of the region.

33. \( R: \text{region bounded by the graphs of } y = 0 \quad \text{and} \quad y = 4 - x^2 \)

34. \( R: \text{region bounded by the graphs of } y = \sqrt{a^2 - x^2} \quad \text{and} \quad y = 0 \)

35. \( R: \text{region bounded by the graphs of } y = x^3 \quad \text{and} \quad y = x, \quad 0 \leq x \leq 1 \)

36. \( R: \text{triangle with vertices } (-a, 0), \quad (a, 0), \quad \text{and} \quad (b, c), \quad \text{where} \quad -a \leq b \leq a \)

Area In Exercises 37–40, use a computer algebra system and the results of Exercise 32 to find the area of the region bounded by the graph of the polar equation.

37. \( r = a(1 - \cos \theta) \)

38. \( r = a \cos 3\theta \)

39. \( r = 1 + 2 \cos \theta \quad \text{(inner loop)} \)

40. \( r = \frac{3}{2 - \cos \theta} \)

41. Think About It Let
   \[ I = \int_C \frac{y \, dx - x \, dy}{x^2 + y^2} \]
   where \( C \) is a circle oriented counterclockwise. Show that \( I = 0 \) if \( C \) does not contain the origin. What is \( I \) if \( C \) contains the origin?
42. (a) Let \( C \) be the line segment joining \((x_1, y_1)\) and \((x_2, y_2)\). Show that
\[
\int_C -y\,dx + x\,dy = x_1 y_2 - x_2 y_1.
\]
(b) Let \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) be the vertices of a polygon. Prove that the area enclosed is
\[
\frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i).
\]

**Area** In Exercises 43 and 44, use the result of Exercise 42(b) to find the area enclosed by the polygon with the given vertices.

43. Pentagon: \((0, 0), (2, 0), (3, 2), (1, 4), (-1, 1)\)
44. Hexagon: \((0, 0), (2, 0), (3, 2), (2, 4), (0, 3), (-1, 1)\)

45. **Investigation** Consider the line integral
\[
\int_C y^n\,dx + x^n\,dy
\]
where \( C \) is the boundary of the region lying between the graphs of \( y = \sqrt{a^2 - x^2} \) \((a > 0)\) and \( y = 0 \).
(a) Use a computer algebra system to verify Green’s Theorem for \( n \), an odd integer from 1 through 7.
(b) Use a computer algebra system to verify Green’s Theorem for \( n \), an even integer from 2 through 8.
(c) For \( n \) an odd integer, make a conjecture about the value of the integral.

46. Green’s first identity:
\[
\iint_D (f \nabla^2 g + \nabla f \cdot \nabla g) \,dA = \oint_C (f \,N \cdot g) \,ds
\]
[Hint: Use the second alternative form of Green’s Theorem and the property \( \text{div} (f \,G) = f \, \text{div} \,G + \nabla f \cdot G \).]

47. Green’s second identity:
\[
\iint_D (f \nabla^2 g - g \nabla f) \,dA = \oint_C (f \,N \cdot g - g \,N \cdot f) \,ds
\]
(Hint: Use Exercise 46 twice.)

48. Use Green’s Theorem to prove that
\[
\int_C f(x)\,dx + g(y)\,dy = 0
\]
if \( f \) and \( g \) are differentiable functions and \( C \) is a piecewise smooth simple closed path.

49. Let \( F = M \,i + N \,j \), where \( M \) and \( N \) have continuous first partial derivatives in a simply connected region \( R \). Prove that if \( C \) is simple, smooth, and closed, and \( N_x = M_y \), then
\[
\oint_C F \cdot \,dr = 0.
\]
Parametric Surfaces

You already know how to represent a curve in the plane or in space by a set of parametric equations—or, equivalently, by a vector-valued function.

\[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{Plane curve} \]
\[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{Space curve} \]

In this section, you will learn how to represent a surface in space by a set of parametric equations—or by a vector-valued function. For curves, note that the vector-valued function \( \mathbf{r} \) is a function of a single parameter \( t \). For surfaces, the vector-valued function is a function of two parameters \( u \) and \( v \).

**Definition of Parametric Surface**

Let \( x, y, \) and \( z \) be functions of \( u \) and \( v \) that are continuous on a domain \( D \) in the \( uv \)-plane. The set of points \((x, y, z)\) given by

\[ \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \]

is called a **parametric surface**. The equations

\[ x = x(u, v), \quad y = y(u, v), \quad \text{and} \quad z = z(u, v) \]

are the **parametric equations** for the surface.

If \( S \) is a parametric surface given by the vector-valued function \( \mathbf{r} \), then \( S \) is traced out by the position vector \( \mathbf{r}(u, v) \) as the point \((u, v)\) moves throughout the domain \( D \), as shown in Figure 15.35.

Figure 15.35

**TECHNOLOGY** Some computer algebra systems are capable of graphing surfaces that are represented parametrically. If you have access to such software, use it to graph some of the surfaces in the examples and exercises in this section.
EXAMPLE 1  Sketching a Parametric Surface

Identify and sketch the parametric surface \( S \) given by
\[
\mathbf{r}(u, v) = 3 \cos u \mathbf{i} + 3 \sin u \mathbf{j} + v \mathbf{k}
\]
where \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 4 \).

**Solution**  Because \( x = 3 \cos u \) and \( y = 3 \sin u \), you know that for each point \((x, y, z)\) on the surface, \( x \) and \( y \) are related by the equation \( x^2 + y^2 = 3^2 \). In other words, each cross section of \( S \) taken parallel to the \( xy \)-plane is a circle of radius 3, centered on the \( z \)-axis. Because \( z = v \), where \( 0 \leq v \leq 4 \), you can see that the surface is a right circular cylinder of height 4. The radius of the cylinder is 3, and the \( z \)-axis forms the axis of the cylinder, as shown in Figure 15.36.

![Figure 15.36](image1.png)

As with parametric representations of curves, parametric representations of surfaces are not unique. That is, there are many other sets of parametric equations that could be used to represent the surface shown in Figure 15.36.

EXAMPLE 2  Sketching a Parametric Surface

Identify and sketch the parametric surface \( S \) given by
\[
\mathbf{r}(u, v) = \sin u \cos v \mathbf{i} + \sin u \sin v \mathbf{j} + \cos u \mathbf{k}
\]
where \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq 2\pi \).

**Solution**  To identify the surface, you can try to use trigonometric identities to eliminate the parameters. After some experimentation, you can discover that
\[
x^2 + y^2 + z^2 = (\sin u \cos v)^2 + (\sin u \sin v)^2 + (\cos u)^2
\]
\[
= \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u
\]
\[
= \sin^2 u (\cos^2 v + \sin^2 v) + \cos^2 u
\]
\[
= \sin^2 u + \cos^2 u
\]
\[
= 1.
\]

So, each point on \( S \) lies on the unit sphere, centered at the origin, as shown in Figure 15.37. For fixed \( u = d_i \), \( \mathbf{r}(u, v) \) traces out latitude circles
\[
x^2 + y^2 = \sin^2 d_i, \quad 0 \leq d_i \leq \pi
\]
that are parallel to the \( xy \)-plane, and for fixed \( v = c_i \), \( \mathbf{r}(u, v) \) traces out longitude (or meridian) half-circles.

![Figure 15.37](image2.png)

NOTE  To convince yourself further that the vector-valued function in Example 2 traces out the entire unit sphere, recall that the parametric equations
\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad \text{and} \quad z = \rho \cos \phi
\]
where \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \pi \), describe the conversion from spherical to rectangular coordinates, as discussed in Section 11.7.
Finding Parametric Equations for Surfaces

In Examples 1 and 2, you were asked to identify the surface described by a given set of parametric equations. The reverse problem—that of writing a set of parametric equations for a given surface—is generally more difficult. One type of surface for which this problem is straightforward, however, is a surface that is given by

You can parametrize such a surface as

\[ r(x, y) = xi + yj + f(x, y)k. \]

**EXAMPLE 3 Representing a Surface Parametrically**

Write a set of parametric equations for the cone given by

\[ z = \sqrt{x^2 + y^2} \]

as shown in Figure 15.38.

**Solution** Because this surface is given in the form \( z = f(x, y) \), you can let \( x \) and \( y \) be the parameters. Then the cone is represented by the vector-valued function

\[ r(x, y) = xi + yj + \sqrt{x^2 + y^2}k \]

where \( (x, y) \) varies over the entire \( xy \)-plane.

**EXAMPLE 4 Representing a Surface of Revolution Parametrically**

Write a set of parametric equations for the surface of revolution obtained by revolving

\[ f(x) = \frac{1}{x}, \quad 1 \leq x \leq 10 \]

about the \( x \)-axis.

**Solution** Use the parameters \( u \) and \( v \) as described above to write

\[ x = u, \quad y = f(u) \cos v, \quad \text{and} \quad z = f(u) \sin v \]

where \( 1 \leq u \leq 10 \) and \( 0 \leq v \leq 2\pi \). The resulting surface is a portion of Gabriel’s Horn, as shown in Figure 15.39.

The surface of revolution in Example 4 is formed by revolving the graph of \( y = f(x) \) about the \( x \)-axis. For other types of surfaces of revolution, a similar parametrization can be used. For instance, to parametrize the surface formed by revolving the graph of \( x = f(z) \) about the \( z \)-axis, you can use

\[ z = u, \quad x = f(u) \cos v, \quad \text{and} \quad y = f(u) \sin v. \]
Normal Vectors and Tangent Planes

Let $S$ be a parametric surface given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

over an open region $D$ such that $x$, $y$, and $z$ have continuous partial derivatives on $D$. The partial derivatives of $\mathbf{r}$ with respect to $u$ and $v$ are defined as

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u, v)\mathbf{i} + \frac{\partial y}{\partial u}(u, v)\mathbf{j} + \frac{\partial z}{\partial u}(u, v)\mathbf{k}$$

and

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u, v)\mathbf{i} + \frac{\partial y}{\partial v}(u, v)\mathbf{j} + \frac{\partial z}{\partial v}(u, v)\mathbf{k}.$$  

Each of these partial derivatives is a vector-valued function that can be interpreted geometrically in terms of tangent vectors. For instance, if $v = v_0$ is held constant, then $\mathbf{r}(u, v_0)$ is a vector-valued function of a single parameter and defines a curve $C_1$ that lies on the surface $S$. The tangent vector to $C_1$ at the point $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ is given by

$$\mathbf{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

as shown in Figure 15.40. In a similar way, if $u = u_0$ is held constant, then $\mathbf{r}(u_0, v)$ is a vector-valued function of a single parameter and defines a curve $C_2$ that lies on the surface $S$. The tangent vector to $C_2$ at the point $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ is given by

$$\mathbf{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$  

If the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$ for any $(u, v)$ in $D$, the surface $S$ is called smooth and will have a tangent plane. Informally, a smooth surface is one that has no sharp points or cusps. For instance, spheres, ellipsoids, and paraboloids are smooth, whereas the cone given in Example 3 is not smooth.

Normal Vector to a Smooth Parametric Surface

Let $S$ be a smooth parametric surface

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

defined over an open region $D$ in the $uv$-plane. Let $(u_0, v_0)$ be a point in $D$. A normal vector at the point

$$(x_0, y_0, z_0) = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$$

given by

$$\mathbf{N} = \mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$  

NOTE Figure 15.40 shows the normal vector $\mathbf{r}_u \times \mathbf{r}_v$. The vector $\mathbf{r}_v \times \mathbf{r}_u$ is also normal to $S$ and points in the opposite direction.
**EXAMPLE 5  Finding a Tangent Plane to a Parametric Surface**

Find an equation of the tangent plane to the paraboloid given by
\[ \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k} \]
at the point \((1, 2, 5)\).

**Solution** The point in the \(uv\)-plane that is mapped to the point \((x, y, z) = (1, 2, 5)\) is \((u, v) = (1, 2)\). The partial derivatives of \(\mathbf{r}\) are
\[ \mathbf{r}_u = i + 2u\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = j + 2v\mathbf{k}. \]
The normal vector is given by
\[ \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2ui - 2vj + \mathbf{k} \]
which implies that the normal vector at \((1, 2, 5)\) is \(\mathbf{r}_u \times \mathbf{r}_v = -2i - 4j + \mathbf{k}\). So, an equation of the tangent plane at \((1, 2, 5)\) is
\[ -2(x - 1) - 4(y - 2) + (z - 5) = 0 \]
\[ -2x - 4y + z = -5. \]
The tangent plane is shown in Figure 15.41.

**Area of a Parametric Surface**

To define the area of a parametric surface, you can use a development that is similar to that given in Section 14.5. Begin by constructing an inner partition of \(D\) consisting of \(n\) rectangles, where the area of the \(i\)th rectangle \(D_i\) is \(\Delta A_i = \Delta u_i \Delta v_i\), as shown in Figure 15.42. In each \(D_i\), let \((u_i, v_i)\) be the point that is closest to the origin. At the point \((x_i, y_i, z_i) = (x(u_i, v_i), y(u_i, v_i), z(u_i, v_i))\) on the surface \(S\), construct a tangent plane \(T_i\). The area of the portion of \(S\) that corresponds to \(D_i\), \(\Delta T_i\), can be approximated by a parallelogram in the tangent plane. That is, \(\Delta T_i \approx \Delta S_i\). So, the surface of \(S\) is given by \(\Sigma \Delta S_i \approx \Sigma \Delta T_i\). The area of the parallelogram in the tangent plane is
\[ \|\Delta u_i \mathbf{r}_u \times \Delta v_i \mathbf{r}_v\| = \|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u_i \Delta v_i \]
which leads to the following definition.

**Area of a Parametric Surface**

Let \(S\) be a smooth parametric surface
\[ \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \]
defined over an open region \(D\) in the \(uv\)-plane. If each point on the surface \(S\) corresponds to exactly one point in the domain \(D\), then the surface area of \(S\) is given by
\[ \text{Surface area} = \iint_S dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| dA \]
where \(\mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k}\) and \(\mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}\).
For a surface $S$ given by $z = f(x, y)$, this formula for surface area corresponds to that given in Section 14.5. To see this, you can parametrize the surface using the vector-valued function

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

defined over the region $R$ in the $xy$-plane. Using

$$\mathbf{r}_x = \mathbf{i} + f_x(x, y)\mathbf{k} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} + f_y(x, y)\mathbf{k}$$

you have

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x, y) \\ 0 & 1 & f_y(x, y) \end{vmatrix} = -f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}$$

and $\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}$. This implies that the surface area of $S$ is

$$\text{Surface area} = \int_R \|\mathbf{r}_x \times \mathbf{r}_y\| \, dA = \int_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} \, dA.$$ 

**EXAMPLE 6 Finding Surface Area**

Find the surface area of the unit sphere given by

$$\mathbf{r}(u, v) = \sin u \cos v \mathbf{i} + \sin u \sin v \mathbf{j} + \cos u \mathbf{k}$$

where the domain $D$ is given by $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

**Solution** Begin by calculating $\mathbf{r}_u$ and $\mathbf{r}_v$.

$$\mathbf{r}_u = \cos u \cos v \mathbf{i} + \sin u \sin v \mathbf{j} - \sin u \mathbf{k}$$

$$\mathbf{r}_v = -\sin u \cos v \mathbf{i} + \sin u \cos v \mathbf{j}$$

The cross product of these two vectors is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u \cos v & \sin u \sin v & -\sin u \\ -\sin u \cos v & \sin u \cos v & 0 \end{vmatrix}$$

$$= \sin^2 u \cos v \mathbf{i} + \sin^2 u \sin v \mathbf{j} + \sin u \cos u \mathbf{k}$$

which implies that

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{[\sin^2 u \cos v]^2 + [\sin^2 u \sin v]^2 + [\sin u \cos u]^2}$$

$$= \sqrt{\sin^4 u + \sin^2 u \cos^2 u}$$

$$= \sin u, \quad \sin u > 0 \text{ for } 0 \leq u \leq \pi$$

Finally, the surface area of the sphere is

$$A = \int_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA = \int_0^{2\pi} \int_0^\pi \sin u \, du \, dv$$

$$= \int_0^{2\pi} 2 \, dv$$

$$= 4\pi.$$
EXAMPLE 7  Finding Surface Area

Find the surface area of the torus given by
\[
r(u, v) = (2 + \cos u) \, \cos v \, i + (2 + \cos u) \, \sin v \, j + \sin u \, k
\]
where the domain \( D \) is given by \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 2\pi \). (See Figure 15.43.)

**Solution**  Begin by calculating \( \mathbf{r}_u \) and \( \mathbf{r}_v \).

\[
\mathbf{r}_u = -\sin u \, \cos v \, i - \sin u \, \sin v \, j + \cos v \, k
\]
\[
\mathbf{r}_v = -(2 + \cos u) \, \sin v \, i + (2 + \cos u) \, \cos v \, j
\]

The cross product of these two vectors is

\[
\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ -\sin u \, \cos v & -\sin u \, \sin v & \cos u \\ -(2 + \cos u) \, \sin v & (2 + \cos u) \, \cos v & 0 \end{vmatrix}
= -(2 + \cos u) \, (\cos v \, \cos u \, i + \sin v \, \cos u \, j + \sin u \, k)
\]

which implies that

\[
||\mathbf{r}_u \times \mathbf{r}_v|| = (2 + \cos u) \sqrt{(\cos v \, \cos u)^2 + (\sin v \, \cos u)^2 + \sin^2 u} = (2 + \cos u) \sqrt{\cos^2 u \, (\cos^2 v + \sin^2 v) + \sin^2 u} = (2 + \cos u) \sqrt{\cos^2 u + \sin^2 u} = 2 + \cos u.
\]

Finally, the surface area of the torus is

\[
A = \int_D ||\mathbf{r}_u \times \mathbf{r}_v|| \, dA = \int_0^{2\pi} \int_0^{2\pi} (2 + \cos u) \, du \, dv
= \int_0^{2\pi} 4\pi \, dv
= 8\pi^2.
\]

**Try It**  Exploration A  Exploration B

If the surface \( S \) is a surface of revolution, you can show that the formula for surface area given in Section 7.4 is equivalent to the formula given in this section. For instance, suppose \( f \) is a nonnegative function such that \( f' \) is continuous over the interval \([a, b]\). Let \( S \) be the surface of revolution formed by revolving the graph of \( f \), where \( a \leq x \leq b \), about the \( x \)-axis. From Section 7.4, you know that the surface area is given by

\[
\text{Surface area} = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx.
\]

To represent \( S \) parametrically, let \( x = u \), \( y = f(u) \, \cos v \), and \( z = f(u) \, \sin v \), where \( a \leq u \leq b \) and \( 0 \leq v \leq 2\pi \). Then,

\[
\mathbf{r}(u, v) = u \, i + f(u) \, \cos v \, j + f(u) \, \sin v \, k.
\]

Try showing that the formula

\[
\text{Surface area} = \int_{D_D} ||\mathbf{r}_u \times \mathbf{r}_v|| \, dA
\]

is equivalent to the formula given above (see Exercise 52).
The symbol \( \boxed{\text{[ ]}} \) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on [S] to view the complete solution of the exercise.

Click on [M] to print an enlarged copy of the graph.

In Exercises 1–4, match the vector-valued function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a) 
(b) 

c) 
(d) 

In Exercises 5–8, find the rectangular equation for the surface by eliminating the parameters from the vector-valued function. Identify the surface and sketch its graph.

5. \( \mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + \frac{v}{2} \mathbf{k} \)
6. \( \mathbf{r}(u, v) = 2 u \cos v \mathbf{i} + 2 u \sin v \mathbf{j} + \frac{1}{2} u^2 \mathbf{k} \)
7. \( \mathbf{r}(u, v) = 2 \cos u \mathbf{i} + v \mathbf{j} + 2 \sin u \mathbf{k} \)
8. \( \mathbf{r}(u, v) = 3 \cos v \cos u \mathbf{i} + 3 \cos v \sin u \mathbf{j} + 5 \sin v \mathbf{k} \)

**Think About It** In Exercises 9–12, determine how the graph of the surface \( s(u, v) \) differs from the graph of \( \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k} \) (see figure) where \( 0 \leq u \leq 2 \) and \( 0 \leq v \leq 2\pi \). (It is not necessary to graph \( s \)).

10. \( s(u, v) = u \cos v \mathbf{i} + u^2 \mathbf{j} + u \sin v \mathbf{k} \) 
   \( 0 \leq u \leq 2, \ 0 \leq v \leq 2\pi \)

11. \( s(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k} \) 
   \( 0 \leq u \leq 3, \ 0 \leq v \leq 2\pi \)

12. \( s(u, v) = 4u \cos v \mathbf{i} + 4u \sin v \mathbf{j} + u^2 \mathbf{k} \) 
   \( 0 \leq u \leq 2, \ 0 \leq v \leq 2\pi \)

In Exercises 13–18, use a computer algebra system to graph the surface represented by the vector-valued function.

13. \( \mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 2u \sin v \mathbf{j} + u^2 \mathbf{k} \) 
   \( 0 \leq u \leq 1, \ 0 \leq v \leq 2\pi \)

14. \( \mathbf{r}(u, v) = 2 \cos v \cos u \mathbf{i} + 4 \cos v \sin u \mathbf{j} + \sin v \mathbf{k} \) 
   \( 0 \leq u \leq 2\pi, \ 0 \leq v \leq 2\pi \)

15. \( \mathbf{r}(u, v) = 2 \sinh u \cos v \mathbf{i} + \sinh u \sin v \mathbf{j} + \cosh u \mathbf{k} \) 
   \( 0 \leq u \leq 2, \ 0 \leq v \leq 2\pi \)

16. \( \mathbf{r}(u, v) = 2u \cos v + 2u \sin v \mathbf{j} + v \mathbf{k} \) 
   \( 0 \leq u \leq 1, \ 0 \leq v \leq 3\pi \)

17. \( \mathbf{r}(u, v) = (u - \sin u) \cos v + (1 - \cos u) \sin v \mathbf{j} + u \mathbf{k} \) 
   \( 0 \leq u \leq \pi, \ 0 \leq v \leq 2\pi \)

18. \( \mathbf{r}(u, v) = \cos^2 u \cos v \mathbf{i} + \sin^3 u \sin v \mathbf{j} + u \mathbf{k} \) 
   \( 0 \leq u \leq \frac{\pi}{2}, \ 0 \leq v \leq 2\pi \)

In Exercises 19–26, find a vector-valued function whose graph is the indicated surface.

19. The plane \( z = y \)
20. The plane \( x + y + z = 6 \)
21. The cylinder \( x^2 + y^2 = 16 \)
22. The cylinder \( 4x^2 + y^2 = 16 \)
23. The cylinder \( z = x^2 \)
24. The ellipsoid \( \frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1 \)
25. The part of the plane \( z = 4 \) that lies inside the cylinder \( x^2 + y^2 = 9 \)
26. The part of the paraboloid \( z = x^2 + y^2 \) that lies inside the cylinder \( x^2 + y^2 = 9 \)

**Surface of Revolution** In Exercises 27–30, write a set of parametric equations for the surface of revolution obtained by revolving the graph of the function about the given axis.

<table>
<thead>
<tr>
<th>Function</th>
<th>Axis of Revolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>27. ( y = \frac{x}{2} ) ( 0 \leq x \leq 6 )</td>
<td>x-axis</td>
</tr>
<tr>
<td>28. ( y = x^{3/2} ) ( 0 \leq x \leq 4 )</td>
<td>x-axis</td>
</tr>
<tr>
<td>29. ( x = \sin z ) ( 0 \leq z \leq \pi )</td>
<td>z-axis</td>
</tr>
<tr>
<td>30. ( z = 4 - y^2 ) ( 0 \leq y \leq 2 )</td>
<td>y-axis</td>
</tr>
</tbody>
</table>
31. \( \mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k}, \quad (1, -1, 1) \)

32. \( \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\mathbf{k}, \quad (1, 1, 1) \)

33. \( \mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 3u\sin v\mathbf{j} + u^2\mathbf{k}, \quad (0, 6, 4) \)

34. \( \mathbf{r}(u, v) = 2u\cosh v\mathbf{i} + 2u\sinh v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}, \quad (-4, 0, 2) \)

Area In Exercises 35–42, find the area of the surface over the given region. Use a computer algebra system to verify your results.

35. The part of the plane \[ \mathbf{r}(u, v) = 2ui - \frac{v}{2}\mathbf{j} + \frac{v}{2}\mathbf{k} \]
   where \( 0 \leq u \leq 2 \) and \( 0 \leq v \leq 1 \)

36. The part of the paraboloid \[ \mathbf{r}(u, v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + ur\mathbf{k} \]
   where \( 0 \leq u \leq 2 \) and \( 0 \leq v \leq 2\pi \)

37. The part of the cylinder \[ \mathbf{r}(u, v) = a\cos u\mathbf{i} + a\sin u\mathbf{j} + v\mathbf{k} \]
   where \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq b \)

38. The sphere \[ \mathbf{r}(u, v) = a\sin u\cos v\mathbf{i} + a\sin u\sin v\mathbf{j} + a\cos u\mathbf{k} \]
   where \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq 2\pi \)

39. The part of the cone \[ \mathbf{r}(u, v) = au\cos v\mathbf{i} + au\sin v\mathbf{j} + au\mathbf{k} \]
   where \( 0 \leq u \leq b \) and \( 0 \leq v \leq 2\pi \)

40. The torus \[ \mathbf{r}(u, v) = (a + b\cos v)\cos u\mathbf{i} + (a + b\cos v)\sin u\mathbf{j} + b\sin v\mathbf{k} \]
   where \( a > b \), \( 0 \leq u \leq 2\pi \), and \( 0 \leq v \leq 2\pi \)

41. The surface of revolution \[ \mathbf{r}(u, v) = \sqrt{a}\cos v\mathbf{i} + \sqrt{a}\sin v\mathbf{j} + ak \]
   where \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq 2\pi \)

42. The surface of revolution \[ \mathbf{r}(u, v) = \sin u\cos v\mathbf{i} + au\mathbf{j} + \sin u\sin v\mathbf{k} \]
   where \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq 2\pi \)

Writing About Concepts

43. Define a parametric surface.

44. Give the double integral that yields the surface area of a parametric surface over an open region \( D \).

45. The four figures are graphs of the surface \[ \mathbf{r}(u, v) = u\mathbf{i} + \sin u\cos v\mathbf{j} + \sin u\sin v\mathbf{k}, \]
   \( 0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq 2\pi. \)

Match each of the four graphs with the point in space from which the surface is viewed. The four points are \((10, 0, 0), (-10, 10, 0), (0, 10, 0), \) and \((10, 10, 10)\).
46. Use a computer algebra system to graph three views of the graph of the vector-valued function
\[ \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}, \quad 0 \leq u \leq \pi, \quad 0 \leq v \leq \pi \]
from the points (10, 0, 0), (0, 0, 10), and (10, 10, 10).

47. Investigation Use a computer algebra system to graph the torus
\[ \mathbf{r}(u, v) = (a + b \cos v) \cos u \mathbf{i} + (a + b \cos v) \sin u \mathbf{j} + b \sin v \mathbf{k} \]
for each set of values of \( a \) and \( b \), where \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 2\pi \). Use the results to describe the effects of \( a \) and \( b \) on the shape of the torus.
(a) \( a = 4, \ b = 1 \)  
(b) \( a = 8, \ b = 1 \)  
(c) \( a = 4, \ b = 2 \)  
(d) \( a = 8, \ b = 3 \)

48. Investigation Consider the function in Exercise 16.
(a) Sketch a graph of the function where \( u \) is held constant at \( u = 1 \). Identify the graph.
(b) Sketch a graph of the function where \( v \) is held constant at \( v = 2\pi/3 \). Identify the graph.
(c) Assume that a surface is represented by the vector-valued function \( \mathbf{r} = \mathbf{r}(u, v) \). What generalization can you make about the graph of the function if one of the parameters is held constant?

49. Surface Area The surface of the dome on a new museum is given by
\[ \mathbf{r}(u, v) = 20 \sin u \cos v \mathbf{i} + 20 \sin u \sin v \mathbf{j} + 20 \cos u \mathbf{k} \]
where \( 0 \leq u \leq \pi/3 \) and \( 0 \leq v \leq 2\pi \) and \( \mathbf{r} \) is in meters. Find the surface area of the dome.

50. Find a vector-valued function for the hyperboloid
\[ x^2 + y^2 - z^2 = 1 \]
and determine the tangent plane at (1, 0, 0).

51. Graph and find the area of one turn of the spiral ramp
\[ \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + 2v \mathbf{k} \]
where \( 0 \leq u \leq 3 \), and \( 0 \leq v \leq 2\pi \).

52. Let \( f \) be a nonnegative function such that \( f' \) is continuous over the interval \([a, b]\). Let \( S \) be the surface of revolution formed by revolving the graph of \( f \), where \( a \leq x \leq b \), about the \( x \)-axis. Let \( x = u, y = f(u) \cos v, \) and \( z = f(u) \sin v \), where \( a \leq u \leq b \) and \( 0 \leq v \leq 2\pi \). Then, \( S \) is represented parametrically by
\[ \mathbf{r}(u, v) = u \mathbf{i} + f(u) \cos v \mathbf{j} + f(u) \sin v \mathbf{k}. \]
Show that the following formulas are equivalent.
Surface area \[ = 2\pi \int_a^b f(u) \sqrt{1 + [f'(u)]^2} \, du \]
Surface area \[ = \int_D \| \mathbf{r}_u \times \mathbf{r}_v \| \, dA \]

53. Open-Ended Project The parametric surfaces
\[ x = 3 + \sin u [7 - \cos(3u - 2v) - 2 \cos(3u + v)] \]
\[ y = 3 + \cos u [7 - \cos(3u - 2v) - 2 \cos(3u + v)] \]
\[ z = \sin(3u - 2v) + 2 \sin(3u + v) \]
where \(-\pi \leq u \leq \pi \) and \(-\pi \leq v \leq \pi \), represent the surface shown below. Try to create your own parametric surface using a computer algebra system.

54. Möbius Strip The surface shown in the figure is called a Möbius Strip and can be represented by the parametric equations
\[ x = (a + u \cos \frac{v}{2}) \cos v, \ y = (a + u \cos \frac{v}{2}) \sin v, \ z = u \sin \frac{v}{2} \]
where \(-1 \leq u \leq 1, \ 0 \leq v \leq 2\pi, \) and \( a = 3 \). Try to graph other Möbius strips for different values of \( a \) using a computer algebra system.
Section 15.6  

Surface Integrals

- Evaluate a surface integral as a double integral.
- Evaluate a surface integral for a parametric surface.
- Determine the orientation of a surface.
- Understand the concept of a flux integral.

Surface Integrals

The remainder of this chapter deals primarily with surface integrals. You will first consider surfaces given by \( z = g(x, y) \). Later in this section you will consider more general surfaces given in parametric form.

Let \( S \) be a surface given by \( z = g(x, y) \) and let \( R \) be its projection onto the \( xy \)-plane, as shown in Figure 15.44. Suppose that \( g \), \( g_x \), and \( g_y \) are continuous at all points in \( R \) and that \( f \) is defined on \( S \). Employing the procedure used to find surface area in Section 14.5, evaluate \( f \) at \( (x_i, y_i, z_i) \) and form the sum

\[
\sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta S_i
\]

where \( \Delta S_i = \sqrt{1 + [g_x(x_i, y_i)]^2 + [g_y(x_i, y_i)]^2} \Delta A_i \). Provided the limit of the above sum as \( \|\Delta\| \) approaches 0 exists, the surface integral of \( f \) over \( S \) is defined as

\[
\int_S f(x, y, z) \, dS = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \, \Delta S_i.
\]

This integral can be evaluated by a double integral.

**Theorem 15.10  Evaluating a Surface Integral**

Let \( S \) be a surface with equation \( z = g(x, y) \) and let \( R \) be its projection onto the \( xy \)-plane. If \( g \), \( g_x \), and \( g_y \) are continuous on \( R \) and \( f \) is continuous on \( S \), then the surface integral of \( f \) over \( S \) is

\[
\int_S f(x, y, z) \, dS = \int_R \int f(x, y, g(x, y)) \sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} \, dA.
\]

For surfaces described by functions of \( x \) and \( z \) (or \( y \) and \( z \)), you can make the following adjustments to Theorem 15.10. If \( S \) is the graph of \( y = g(x, z) \) and \( R \) is its projection onto the \( xz \)-plane, then

\[
\int_S f(x, y, z) \, dS = \int_R \int f(x, g(x, z), z) \sqrt{1 + [g_y(x, z)]^2 + [g_z(x, z)]^2} \, dA.
\]

If \( S \) is the graph of \( x = g(y, z) \) and \( R \) is its projection onto the \( yz \)-plane, then

\[
\int_S f(x, y, z) \, dS = \int_R \int f(g(y, z), y, z) \sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2} \, dA.
\]

If \( f(x, y, z) = 1 \), the surface integral over \( S \) yields the surface area of \( S \). For instance, suppose the surface \( S \) is the plane given by \( z = x \), where \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \). The surface area of \( S \) is \( \sqrt{2} \) square units. Try verifying that \( \int_S f(x, y, z) \, dS = \sqrt{2} \).
**Example 1**  Evaluating a Surface Integral

Evaluate the surface integral
\[
\int_S (y^2 + 2yz) \, dS
\]
where \( S \) is the first-octant portion of the plane \( 2x + y + 2z = 6 \).

**Solution**  Begin by writing \( S \) as
\[
z = \frac{1}{2} (6 - 2x - y)
\]
\[
g(x, y) = \frac{1}{2} (6 - 2x - y).
\]

Using the partial derivatives \( g_x(x, y) = -1 \) and \( g_y(x, y) = -\frac{1}{2} \), you can write
\[
\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} = \sqrt{1 + 1 + \frac{1}{4}} = \frac{3}{2}.
\]

Using Figure 15.45 and Theorem 15.10, you obtain
\[
\int_S (y^2 + 2yz) \, dS = \int_R \int f(x, y, g(x, y)) \sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} \, dA
\]
\[
= \int_R \int [y^2 + 2y(\frac{1}{2})(6 - 2x - y)] (\frac{3}{2}) \, dA
\]
\[
= 3 \int_0^3 \int_0^{2(3-x)} y(3 - x) \, dy \, dx
\]
\[
= 6 \int_0^3 (3 - x)^3 \, dx
\]
\[
= -\frac{3}{2} (3 - x)^4 \bigg|_0^3
\]
\[
= \frac{243}{2}.
\]

**Try It**

An alternative solution to Example 1 would be to project \( S \) onto the \( yz \)-plane, as shown in Figure 15.46. Then, \( x = \frac{1}{2} (6 - y - 2z) \), and
\[
\sqrt{1 + [g_y(x, z)]^2 + [g_z(x, z)]^2} = \sqrt{1 + \frac{1}{4}} + 1 = \frac{3}{2}.
\]

So, the surface integral is
\[
\int_S (y^2 + 2yz) \, dS = \int_R \int f(g(y, z), y, z) \sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2} \, dA
\]
\[
= \int_0^6 \int_{(6-y)/2}^{(6-y)/2} (y^2 + 2yz) (\frac{3}{2}) \, dz \, dy
\]
\[
= \frac{3}{8} \int_0^6 (36y - y^3) \, dy
\]
\[
= \frac{243}{2}.
\]

Try reworking Example 1 by projecting \( S \) onto the \( xz \)-plane.
In Example 1, you could have projected the surface $S$ onto any one of the three coordinate planes. In Example 2, $S$ is a portion of a cylinder centered about the $x$-axis, and you can project it onto either the $xz$-plane or the $xy$-plane.

**Example 2  Evaluating a Surface Integral**

Evaluate the surface integral

$$
\int_S (x + z) \, dS
$$

where $S$ is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between $x = 0$ and $x = 4$, as shown in Figure 15.47.

**Solution** Project $S$ onto the $xy$-plane, so that $z = g(x, y) = \sqrt{9 - y^2}$, and obtain

$$
\sqrt{1 + \left[g_x(x, y)\right]^2 + \left[g_y(x, y)\right]^2} = \sqrt{1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2} = \frac{3}{\sqrt{9 - y^2}}.
$$

Theorem 15.10 does not apply directly because $g_y$ is not continuous when $y = 3$. However, you can apply the theorem for $0 \leq b < 3$ and then take the limit as $b$ approaches 3, as follows.

$$
\int_S (x + z) \, dS = \lim_{b \to 3^-} \int_0^b \int_0^4 \left(x + \sqrt{9 - y^2}\right) \frac{3}{\sqrt{9 - y^2}} \, dx \, dy
$$

$$
= \lim_{b \to 3^-} \frac{3}{2} \int_0^4 \left(\frac{x^2}{9 - y^2} + 1\right) \, dx \, dy
$$

$$
= \lim_{b \to 3^-} \frac{3}{2} \int_0^4 \left(\frac{8}{9 - y^2} + 4\right) \, dy
$$

$$
= \int_0^3 \left(4b + 8 \arcsin b\right) \, dy
$$

$$
= 36 + 24 \left(\frac{\pi}{2}\right)
$$

$$
= 36 + 12\pi
$$

**TECHNOLOGY** Some computer algebra systems are capable of evaluating improper integrals. If you have access to such computer software, use it to evaluate the improper integral

$$
\int_0^3 \int_0^4 \left(x + \sqrt{9 - y^2}\right) \frac{3}{\sqrt{9 - y^2}} \, dx \, dy.
$$

Do you obtain the same result as in Example 2?
You have already seen that if the function defined on the surface is simply $f(x, y, z) = 1$, the surface integral yields the surface area of $S$.

$$\text{Area of surface} = \iint_S 1 \, dS$$

On the other hand, if $S$ is a lamina of variable density and $\rho(x, y, z)$ is the density at the point $(x, y, z)$, then the mass of the lamina is given by

$$\text{Mass of lamina} = \iint_S \rho(x, y, z) \, dS.$$  

**EXAMPLE 3  Finding the Mass of a Surface Lamina**

A cone-shaped surface lamina $S$ is given by

$$z = 4 - 2\sqrt{x^2 + y^2}, \quad 0 \leq z \leq 4$$

as shown in Figure 15.48. At each point on $S$, the density is proportional to the distance between the point and the $z$-axis. Find the mass $m$ of the lamina.

**Solution** Projecting $S$ onto the $xy$-plane produces

$$S: z = 4 - 2\sqrt{x^2 + y^2} = g(x, y), \quad 0 \leq z \leq 4$$

$$R: x^2 + y^2 \leq 4$$

with a density of $\rho(x, y, z) = k\sqrt{x^2 + y^2}$. Using a surface integral, you can find the mass to be

$$m = \iint_S \rho(x, y, z) \, dS$$

$$= \int_R \int k\sqrt{x^2 + y^2}\sqrt{1 + \left[g_x(x, y)\right]^2 + \left[g_y(x, y)\right]^2} \, dA$$

$$= k\int_R \int \sqrt{x^2 + y^2}\sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} \, dA$$

$$= k\int_R \int \sqrt{5}\sqrt{x^2 + y^2} \, dA$$

$$= k\int_0^{2\pi} \int_0^2 (\sqrt{5}r) r \, dr \, d\theta$$

$$= \frac{\sqrt{5}k}{3} \left[ \frac{r^3}{3} \right]_0^2 \left[ \frac{2\pi}{3} \right]$$

$$= \frac{8\sqrt{5}k}{3} \left[ \frac{2\pi}{3} \right] = \frac{16\sqrt{5}k\pi}{3}.$$

**TECHNOLOGY** Use a computer algebra system to confirm the result shown in Example 3. The computer algebra system Derive evaluated the integral as follows.

$$k\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{5}\sqrt{x^2 + y^2} \, dx \, dy = \frac{16\sqrt{5}k\pi}{3}.$$
**Parametric Surfaces and Surface Integrals**

For a surface \( S \) given by the vector-valued function

\[
r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k
\]

defined over a region \( D \) in the \( uv\)-plane, you can show that the surface integral of \( f(x, y, z) \) over \( S \) is given by

\[
\int_S f(x, y, z) \, dS = \int_D f(x(u, v), y(u, v), z(u, v)) \left\| r_u(u, v) \times r_v(u, v) \right\| \, dA.
\]

Note the similarity to a line integral over a space curve \( C \).

\[
\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \left\| \mathbf{r}'(t) \right\| \, dt
\]

Line integral

**EXAMPLE 4  Evaluating a Surface Integral**

Example 2 demonstrated an evaluation of the surface integral

\[
\int_S (x + z) \, dS
\]

where \( S \) is the first-octant portion of the cylinder \( y^2 + z^2 = 9 \) between \( x = 0 \) and \( x = 4 \) (see Figure 15.49). Reevaluate this integral in parametric form.

**Solution**  In parametric form, the surface is given by

\[
r(x, \theta) = xi + 3 \cos \theta j + 3 \sin \theta k
\]

where \( 0 \leq x \leq 4 \) and \( 0 \leq \theta \leq \pi/2 \). To evaluate the surface integral in parametric form, begin by calculating the following.

\[
\mathbf{r}_x = \mathbf{i}
\]

\[
\mathbf{r}_\theta = -3 \sin \theta \mathbf{j} + 3 \cos \theta \mathbf{k}
\]

\[
\mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & -3 \sin \theta & 3 \cos \theta \end{vmatrix} = -3 \cos \theta \mathbf{j} - 3 \sin \theta \mathbf{k}
\]

\[
\left\| \mathbf{r}_x \times \mathbf{r}_\theta \right\| = \sqrt{9 \cos^2 \theta + 9 \sin^2 \theta} = 3
\]

So, the surface integral can be evaluated as follows.

\[
\int_D (x + 3 \sin \theta) \, dA = \int_0^4 \int_0^{\pi/2} (3x + 9 \sin \theta) \, d\theta \, dx
\]

\[
= \int_0^4 \left[ 3x\theta - 9 \cos \theta \right]_0^{\pi/2} \, dx
\]

\[
= \int_0^4 \left( \frac{3\pi}{2} - 9 \right) \, dx
\]

\[
= \left[ \frac{3\pi}{4} x^2 + 9x \right]_0^4
\]

\[
= 12\pi + 36
\]

**Try It**  **Exploration A**
Orientation of a Surface

Unit normal vectors are used to induce an orientation to a surface \( S \) in space. A surface is called \textbf{orientable} if a unit normal vector \( \mathbf{N} \) can be defined at every nonboundary point of \( S \) in such a way that the normal vectors vary continuously over the surface \( S \). If this is possible, \( S \) is called an \textbf{oriented surface}.

An orientable surface \( S \) has two distinct sides. So, when you orient a surface, you are selecting one of the two possible unit normal vectors. If \( S \) is a closed surface such as a sphere, it is customary to choose the unit normal vector \( \mathbf{N} \) to be the one that points outward from the sphere.

Most common surfaces, such as spheres, paraboloids, ellipses, and planes, are orientable. (See Exercise 43 for an example of a surface that is \textit{not} orientable.) Moreover, for an orientable surface, the gradient vector provides a convenient way to find a unit normal vector. That is, for an orientable surface \( S \) given by

\[
z = g(x, y)
\]

let

\[
G(x, y, z) = z - g(x, y).
\]

Then, \( S \) can be oriented by either the unit normal vector

\[
\mathbf{N} = \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}
\]

\[
= \frac{-g_y(x, y)\mathbf{i} - g_z(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{1 + [g_y(x, y)]^2 + [g_z(x, y)]^2}} \quad \text{Upward unit normal}
\]

or the unit normal vector

\[
\mathbf{N} = -\frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}
\]

\[
= \frac{g_y(x, y)\mathbf{i} + g_z(x, y)\mathbf{j} - \mathbf{k}}{\sqrt{1 + [g_y(x, y)]^2 + [g_z(x, y)]^2}} \quad \text{Downward unit normal}
\]

as shown in Figure 15.50. If the smooth orientable surface \( S \) is given in parametric form by

\[
\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \quad \text{Parametric surface}
\]

the unit normal vectors are given by

\[
\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}
\]

and

\[
\mathbf{N} = \frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_v \times \mathbf{r}_u\|}
\]

\textbf{NOTE} Suppose that the orientable surface is given by \( y = g(x, z) \) or \( x = g(y, z) \). Then you can use the gradient vector

\[
\nabla G(x, y, z) = -g_z(x, z)\mathbf{i} + \mathbf{j} - g_y(x, z)\mathbf{k} \quad G(x, y, z) = y - g(x, z)
\]

or

\[
\nabla G(x, y, z) = \mathbf{i} - g_y(y, z)\mathbf{j} - g_z(y, z)\mathbf{k} \quad G(x, y, z) = x - g(y, z)
\]

to orient the surface.
Flux Integrals

One of the principal applications involving the vector form of a surface integral relates to the flow of a fluid through a surface \( S \). Suppose an oriented surface \( S \) is submerged in a fluid having a continuous velocity field \( \mathbf{F} \). Let \( \Delta S \) be the area of a small patch of the surface \( S \) over which \( \mathbf{F} \) is nearly constant. Then the amount of fluid crossing this region per unit of time is approximated by the volume of the column of height \( \mathbf{F} \cdot \mathbf{N} \), as shown in Figure 15.51. That is,

\[
\Delta V = (\text{height})(\text{area of base}) = (\mathbf{F} \cdot \mathbf{N})\Delta S.
\]

Consequently, the volume of fluid crossing the surface \( S \) per unit of time (called the flux of \( \mathbf{F} \) across \( S \)) is given by the surface integral in the following definition.

**Definition of Flux Integral**

Let \( \mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + P \mathbf{k} \), where \( M, N, \) and \( P \) have continuous first partial derivatives on the surface \( S \) oriented by a unit normal vector \( \mathbf{N} \). The flux integral of \( \mathbf{F} \) across \( S \) is given by

\[
\iint_S \mathbf{F} \cdot \mathbf{N} \, dS.
\]

Geometrically, a flux integral is the surface integral over \( S \) of the normal component of \( \mathbf{F} \). If \( \rho(x, y, z) \) is the density of the fluid at \((x, y, z)\), the flux integral

\[
\iint_S \rho \mathbf{F} \cdot \mathbf{N} \, dS
\]

represents the mass of the fluid flowing across \( S \) per unit of time.

To evaluate a flux integral for a surface given by \( z = g(x, y) \), let

\[
G(x, y, z) = z - g(x, y).
\]

Then, \( \mathbf{N} \, dS \) can be written as follows.

\[
\mathbf{N} \, dS = \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|} \, dS
= \frac{\nabla G(x, y, z)}{\sqrt{(g_x)^2 + (g_y)^2 + 1}} \sqrt{(g_x)^2 + (g_y)^2 + 1} \, dA
= \nabla G(x, y, z) \, dA
\]

**THEOREM 15.11 Evaluating a Flux Integral**

Let \( S \) be an oriented surface given by \( z = g(x, y) \) and let \( R \) be its projection onto the \( xy \)-plane.

\[
\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \int_R \int \mathbf{F} \cdot [\mathbf{N} \, dS]
= \iint_R \mathbf{F} \cdot [g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}] \, dA \quad \text{Oriented upward}
\]

\[
\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \int_R \int \mathbf{F} \cdot [\mathbf{N} \, dS]
= \iint_R \mathbf{F} \cdot [g_x(x, y)i + g_y(x, y)j - \mathbf{k}] \, dA \quad \text{Oriented downward}
\]

For the first integral, the surface is oriented upward, and for the second integral, the surface is oriented downward.
EXAMPLE 5  Using a Flux Integral to Find the Rate of Mass Flow

Let \( S \) be the portion of the paraboloid 
\[
z = g(x, y) = 4 - x^2 - y^2
\]
lying above the \( xy \)-plane, oriented by an upward unit normal vector, as shown in Figure 15.52. A fluid of constant density \( \rho \) is flowing through the surface \( S \) according to the vector field 
\[
\mathbf{F}(x, y, z) = xi + yj + zk.
\]
Find the rate of mass flow through \( S \).

Solution  Begin by computing the partial derivatives of \( g \).
\[
g_x(x, y) = -2x
\]
and
\[
g_y(x, y) = -2y
\]
The rate of mass flow through the surface \( S \) is 
\[
\int_S \rho \mathbf{F} \cdot \mathbf{N} \, dS = \rho \int_R \left[ \mathbf{F} \cdot \left( -g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k} \right) \right] \, dA
\]
\[
= \rho \int_R \left[ x\mathbf{i} + y\mathbf{j} + (4 - x^2 - y^2)\mathbf{k} \right] \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) \, dA
\]
\[
= \rho \int_R \left[ 2x^2 + 2y^2 + (4 - x^2 - y^2) \right] \, dA
\]
\[
= \rho \int_0^{2\pi} \int_0^2 (4 + r^2) \, r \, dr \, d\theta \quad \text{Polar coordinates}
\]
\[
= \rho \int_0^{2\pi} 12 \, d\theta
\]
\[
= 24\pi\rho.
\]

Try It  Exploration A

For an oriented surface \( S \) given by the vector-valued function 
\[
\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}
\]
defined over a region \( D \) in the \( uv \)-plane, you can define the flux integral of \( \mathbf{F} \) across \( S \) as
\[
\int_S \mathbf{F} \cdot \mathbf{N} \, dS = \int_D \left[ \mathbf{F} \cdot \left( \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \right) \|\mathbf{r}_u \times \mathbf{r}_v\| \right] \, dA
\]
\[
= \int_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.
\]
Note the similarity of this integral to the line integral 
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds.
\]
A summary of formulas for line and surface integrals is presented on page 1117.
**Example 6** Finding the Flux of an Inverse Square Field

Find the flux over the sphere \( S \) given by
\[
x^2 + y^2 + z^2 = a^2
\]
where \( F \) is an inverse square field given by
\[
F(x, y, z) = \frac{kr}{|r|^3} = \frac{kqr}{|r|^3}
\]
and \( r = xi + yj + zk \). Assume \( S \) is oriented outward, as shown in Figure 15.53.

**Solution** The sphere is given by
\[
r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k
\]
where \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq 2\pi \). The partial derivatives of \( r \) are
\[
r_u(u, v) = a \cos u \cos v \ i + a \sin u \sin v \ j + a \cos u \ k
\]
and
\[
r_v(u, v) = -a \sin u \ j + a \sin u \ cos \ v \ k
\]
which implies that the normal vector \( r_u \times r_v \) is
\[
r_u \times r_v = \begin{vmatrix}
  i & j & k \\
  a \cos u \cos v & a \cos u \sin v & -a \sin u \\
  a \sin u \sin v & a \sin u \cos v & 0
\end{vmatrix}
\]
\[
= a^2 \begin{pmatrix}
  \sin^2 u \cos v \ i + \sin^2 u \sin v \ j + \sin u \cos u \ k
\end{pmatrix}
\]
Now, using
\[
F(x, y, z) = \frac{kqr}{|r|^3}
\]
\[
= kq \frac{xi + yj + zk}{|x + yj + zk|^3}
\]
\[
= \frac{kq}{a^3}(a \sin u \cos vi + a \sin u \sin vj + a \cos uk)
\]
it follows that
\[
F \cdot (r_u \times r_v) = \frac{kq}{a^3}[(a \sin u \cos vi + a \sin u \sin vj + a \cos uk) \cdot
\]
\[
(a^2 \sin^2 u \cos v + \sin^2 u \sin v + \sin u \cos u)]
\]
\[
= kq(\sin^3 u \cos^2 v + \sin^3 u \sin^2 v + \sin u \cos^2 u)
\]
\[
= kq \sin u.
\]
Finally, the flux over the sphere \( S \) is given by
\[
\int_S \int F \cdot N \ dS = \int_0^{2\pi} \int_0^\pi (kq \sin u) \ du \ dv
\]
\[
= \int_0^{2\pi} \int_0^\pi kq \sin u \ du \ dv
\]
\[
= 4\pi kq.
\]
The result in Example 6 shows that the flux across a sphere in an inverse square field is independent of the radius of the sphere. In particular, if \( \mathbf{E} \) is an electric field, the result in Example 6, along with Coulomb’s Law, yields one of the basic laws of electrostatics, known as Gauss’s Law:

\[
\int_S \mathbf{E} \cdot \mathbf{N} \, dS = 4\pi k q
\]

Gauss’s Law

where \( q \) is a point charge located at the center of the sphere and \( k \) is the Coulomb constant. Gauss’s Law is valid for more general closed surfaces that enclose the origin, and relates the flux out of the surface to the total charge \( q \) inside the surface.

This section concludes with a summary of different forms of line integrals and surface integrals.

### Summary of Line and Surface Integrals

#### Line Integrals

\[
ds = \|\mathbf{r}'(t)\| \, dt
\]

\[
= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt
\]

\[
\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \, ds
\]

Scalar form

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds
\]

\[
= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) \, dt
\]

Vector form

#### Surface Integrals (parametric form)

\[
ds = \|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)\| \, dA
\]

\[
\int_S f(x, y, z) \, dS = \int_D \int f(x(u, v), y(u, v), z(u, v)) \, dS
\]

Scalar form

\[
\int_S \mathbf{F} \cdot \mathbf{N} \, dS = \int_D \int \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA
\]

Vector form

#### Surface Integrals (scalar form)

\[
ds = \sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} \, dA
\]

\[
\int_S f(x, y, z) \, dS = \int_R \int f(x, y, g(x, y)) \sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} \, dA
\]

Scalar form

\[
\int_S \mathbf{F} \cdot \mathbf{N} \, dS = \int_R \int \mathbf{F} \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}] \, dA
\]

Vector form (upward normal)
Exercises for Section 15.6

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on to view the complete solution of the exercise.

Click on to print an enlarged copy of the graph.

In Exercises 1–4, evaluate \( \int_S (x - 2y + z) \, dS \).

1. \( S: z = 4 - x, \quad 0 \leq x \leq 4, \quad 0 \leq y \leq 4 \)
2. \( S: z = 15 - 2x + 3y, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 4 \)
3. \( S: z = 10, \quad x^2 + y^2 \leq 1 \)
4. \( S: z = \frac{1}{2}x^{3/2}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq x \)

In Exercises 5 and 6, evaluate \( \int_S xy \, dS \).

5. \( S: z = 6 - x - 2y, \) first octant
6. \( S: z = h, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq \sqrt{4 - x^2} \)

In Exercises 7 and 8, use a computer algebra system to evaluate \( \int_S xy \, dS \).

7. \( S: z = 9 - x^2, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq x \)
8. \( S: z = \frac{1}{2}xy, \quad 0 \leq x \leq 4, \quad 0 \leq y \leq 4 \)

In Exercises 9 and 10, use a computer algebra system to evaluate \( \int_S (x^2 - 2xy) \, dS \).

9. \( S: z = 10 - x^2 - y^2, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2 \)
10. \( S: z = \cos x, \quad 0 \leq x \leq \frac{\pi}{2}, \quad 0 \leq y \leq \frac{1}{2}x \)

Mass In Exercises 11 and 12, find the mass of the surface lamina \( S \) of density \( \rho \).

11. \( S: 2x + 3y + 6z = 12, \) first octant, \( \rho(x, y, z) = x^2 + y^2 \)
12. \( S: z = \sqrt{a^2 - x^2 - y^2}, \quad \rho(x, y, z) = k \)

In Exercises 13–16, evaluate \( \int_S f(x, y) \, dS \).

13. \( f(x, y) = y + 5 \)
   \( S: \mathbf{r}(u, v) = ui + vj + \frac{u}{2}k, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2 \)
14. \( f(x, y) = x + y \)
   \( S: \mathbf{r}(u, v) = 2\cos u i + 2\sin u j + v k \)
   \( 0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq 2 \)
15. \( f(x, y) = xy \)
   \( S: \mathbf{r}(u, v) = 2\cos u i + 2\sin u j + v k \)
   \( 0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq 2 \)
16. \( f(x, y) = x + y \)
   \( S: \mathbf{r}(u, v) = 4u \cos v i + 4u \sin v j + 3uk \)
   \( 0 \leq u \leq 4, \quad 0 \leq v \leq \pi \)

In Exercises 17–22, evaluate \( \int_S f(x, y, z) \, dS \).

17. \( f(x, y, z) = x^2 + y^2 + z^2 \)
   \( S: z = x + 2, \quad x^2 + y^2 \leq 1 \)
18. \( f(x, y, z) = \frac{3y}{z} \)
   \( S: z = x^2 + y^2, \quad 4 \leq x^2 + y^2 \leq 16 \)
19. \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \)
   \( S: z = \sqrt{x^2 + y^2}, \quad x^2 + y^2 \leq 4 \)
20. \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \)
   \( S: z = \sqrt{x^2 + y^2}, \quad (x - 1)^2 + y^2 \leq 1 \)
21. \( f(x, y, z) = x^2 + y^2 + z^2 \)
   \( S: x^2 + y^2 = 9, \quad 0 \leq x \leq 3, \quad 0 \leq y \leq 3, \quad 0 \leq z \leq 9 \)
22. \( f(x, y, z) = x^2 + y^2 + z^2 \)
   \( S: x^2 + y^2 = 9, \quad 0 \leq x \leq 3, \quad 0 \leq z \leq x \)

In Exercises 23–28, find the flux of \( F \) through \( S \), \( \int_S F \cdot \mathbf{N} \, dS \) where \( N \) is the upward unit normal vector to \( S \).

23. \( F(x, y, z) = 3zi - 4j + yk \)
   \( S: x + y + z = 1, \) first octant
24. \( F(x, y, z) = xi + yj \)
   \( S: 2x + 3y + z = 6, \) first octant
25. \( F(x, y, z) = xi + yj + zk \)
   \( S: \quad z = 9 - x^2 - y^2, \quad z \geq 0 \)
26. \( F(x, y, z) = xi + yj + zk \)
   \( S: x^2 + y^2 + z^2 = 36, \) first octant
27. \( F(x, y, z) = 4i - 3j + 5k \)
   \( S: z = x^2 + y^2, \quad x^2 + y^2 \leq 4 \)
28. \( F(x, y, z) = xi + yj - 2k \)
   \( S: z = \sqrt{a^2 - x^2 - y^2} \)

In Exercises 29 and 30, find the flux of \( F \) over the closed surface (Let \( N \) be the outward unit normal vector of the surface.)

29. \( F(x, y, z) = 4xyi + z^2j + yzk \)
   \( S: \) unit cube bounded by \( x = 0, \) \( x = 1, \) \( y = 0, \) \( y = 1, \) \( z = 0, \) \( z = 1 \)
30. \( F(x, y, z) = (x + y)i + yj + zk \)
   \( S: z = 1 - x^2 - y^2, \quad z = 0 \)
31. Define a surface integral of the scalar function $f$ over a surface $z = g(x, y)$. Explain how to evaluate the surface integral.

32. Describe an orientable surface.

33. Define a flux integral and explain how it is evaluated.

34. Is the surface shown in the figure orientable? Explain.

35. Electrical Charge Let $E = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ be an electrostatic field. Use Gauss’s Law to find the total charge enclosed by the closed surface consisting of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and its circular base in the $xy$-plane.

36. Electrical Charge Let $E = xi + yj + zk$ be an electrostatic field. Use Gauss’s Law to find the total charge enclosed by the closed surface consisting of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and its circular base in the $xy$-plane.

Moment of Inertia In Exercises 37 and 38, use the following formulas for the moments of inertia about the coordinate axes of a surface lamina of density $\rho$.

$$I_x = \iint_S (y^2 + z^2)\rho(x, y, z)\,dS$$
$$I_y = \iint_S (x^2 + z^2)\rho(x, y, z)\,dS$$
$$I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\,dS$$

37. Verify that the moment of inertia of a conical shell of uniform density about its axis is $\frac{3}{2}ma^2$, where $m$ is the mass and $a$ is the radius and height.

38. Verify that the moment of inertia of a spherical shell of uniform density about its diameter is $\frac{3}{8}ma^2$, where $m$ is the mass and $a$ is the radius.

Flow Rate In Exercises 41 and 42, use a computer algebra system to find the rate of mass flow of a fluid of density $\rho$ through the surface $S$ oriented upward if the velocity field $\mathbf{v}$ given by $F(x, y, z) = 0.5\mathbf{k}$.

41. $S: z = 16 - x^2 - y^2, \quad z \geq 0$

42. $S: z = \sqrt{16 - x^2 - y^2}$

43. Investigation

(a) Use a computer algebra system to graph the vector-value function

$$\mathbf{r}(u, v) = (4 - v \sin u) \cos(2u)\mathbf{i} + (4 - v \sin u) \sin(2u)\mathbf{j} + v \cos u\mathbf{k}, \quad 0 \leq u \leq \pi, \quad -1 \leq v \leq 1.$$ 

This surface is called a Möbius strip.

(b) Explain why this surface is not orientable.

(c) Use a computer algebra system to graph the space curve represented by $\mathbf{r}(u, 0)$. Identify the curve.

(d) Construct a Möbius strip by cutting a strip of paper, making a single twist, and pasting the ends together.

(e) Cut the Möbius strip along the space curve graphed in part (c), and describe the result.
Divergence Theorem

Recall from Section 15.4 that an alternative form of Green’s Theorem is
\[
\int_C F \cdot N \, ds = \int_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dA.
\]
\[
= \int_R \text{div} \, F \, dA.
\]

In an analogous way, the **Divergence Theorem** gives the relationship between a triple integral over a solid region \( Q \) and a surface integral over the surface of \( Q \). In the statement of the theorem, the surface \( S \) is **closed** in the sense that it forms the complete boundary of the solid \( Q \). Regions bounded by spheres, ellipsoids, cubes, tetrahedrons, or combinations of these surfaces are typical examples of closed surfaces. Assume that \( Q \) is a solid region on which a triple integral can be evaluated, and that the closed surface \( S \) is oriented by **outward** unit normal vectors, as shown in Figure 15.54. With these restrictions on \( S \) and \( Q \), the Divergence Theorem is as follows.

**Figure 15.54**

**THEOREM 15.12** **The Divergence Theorem**

Let \( Q \) be a solid region bounded by a closed surface \( S \) oriented by a unit normal vector directed outward from \( Q \). If \( F \) is a vector field whose component functions have continuous partial derivatives in \( Q \), then
\[
\iiint_Q \text{div} \, F \, dV = \iint_S F \cdot N \, dS.
\]

**NOTE** As noted at the left above, the Divergence Theorem is sometimes called Gauss’s Theorem. It is also sometimes called Ostrogradsky’s Theorem, after the Russian mathematician Michel Ostrogradsky (1801–1861).
Proof. If you let \( \mathbf{F}(x, y, z) = M \mathbf{i} + N \mathbf{j} + P \mathbf{k} \), the theorem takes the form

\[
\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q (M \mathbf{i} \cdot \mathbf{N} + N \mathbf{j} \cdot \mathbf{N} + P \mathbf{k} \cdot \mathbf{N}) \, dS = \iiint_Q \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) \, dV.
\]

You can prove this by verifying that the following three equations are valid.

\[
\begin{align*}
\iint_S M \mathbf{i} \cdot \mathbf{N} \, dS &= \iiint_Q \frac{\partial M}{\partial x} \, dV \\
\iint_S N \mathbf{j} \cdot \mathbf{N} \, dS &= \iiint_Q \frac{\partial N}{\partial y} \, dV \\
\iint_S P \mathbf{k} \cdot \mathbf{N} \, dS &= \iiint_Q \frac{\partial P}{\partial z} \, dV
\end{align*}
\]

Because the verifications of the three equations are similar, only the third is discussed. Restrict the proof to a simple solid region with upper surface

\[ z = g_2(x, y) \quad \text{Upper surface} \]

and lower surface

\[ z = g_1(x, y) \quad \text{Lower surface} \]

whose projections onto the \( xy \)-plane coincide and form region \( R \). If \( Q \) has a lateral surface \( S_1 \) in Figure 15.55, then a normal vector is horizontal, which implies that \( P \mathbf{k} \cdot \mathbf{N} = 0 \). Consequently, you have

\[
\iint_S P \mathbf{k} \cdot \mathbf{N} \, dS = \iint_{S_1} P \mathbf{k} \cdot \mathbf{N} \, dS + \iint_{S_2} P \mathbf{k} \cdot \mathbf{N} \, dS + 0.
\]

On the upper surface \( S_2 \), the outward normal vector is upward, whereas on the lower surface \( S_1 \), the outward normal vector is downward. So, by Theorem 15.11, you have the following.

\[
\begin{align*}
\iint_{S_1} P \mathbf{k} \cdot \mathbf{N} \, dS &= \iint_{R} P(x, y, g_1(x, y)) \left( \frac{\partial g_1}{\partial x} \mathbf{i} + \frac{\partial g_1}{\partial y} \mathbf{j} - \mathbf{k} \right) \, dA \\
&= -\iint_{R} P(x, y, g_1(x, y)) \, dA \\
\iint_{S_2} P \mathbf{k} \cdot \mathbf{N} \, dS &= \iint_{R} P(x, y, g_2(x, y)) \left( -\frac{\partial g_2}{\partial x} \mathbf{i} - \frac{\partial g_2}{\partial y} \mathbf{j} + \mathbf{k} \right) \, dA \\
&= \iint_{R} P(x, y, g_2(x, y)) \, dA
\end{align*}
\]

Adding these results, you obtain

\[
\begin{align*}
\iint_S P \mathbf{k} \cdot \mathbf{N} \, dS &= \iint_{R} \left[ P(x, y, g_2(x, y)) - P(x, y, g_1(x, y)) \right] \, dA \\
&= \iint_{R} \left[ \int_{g_1(x, y)}^{g_2(x, y)} \frac{\partial P}{\partial z} \, dz \right] \, dA \\
&= \iiint_Q \frac{\partial P}{\partial z} \, dV.
\end{align*}
\]
EXAMPLE 1 Using the Divergence Theorem

Let $Q$ be the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$, and let $\mathbf{F} = xi + y^2j + zk$. Find

$$\int_S \mathbf{F} \cdot \mathbf{N} \, dS$$

where $S$ is the surface of $Q$.

Solution From Figure 15.56, you can see that $Q$ is bounded by four subsurfaces. So, you would need four surface integrals to evaluate

$$\int_S \mathbf{F} \cdot \mathbf{N} \, dS.$$

However, by the Divergence Theorem, you need only one triple integral. Because

$$\text{div} \, \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= 1 + 2y + 1$$

$$= 2 + 2y$$

you have

$$\int_S \mathbf{F} \cdot \mathbf{N} \, dS = \int_Q \int \int \text{div} \, \mathbf{F} \, dV$$

$$= \int_0^3 \int_0^{3-y} \int_0^{6-2x-2y} (2 + 2y) \, dz \, dx \, dy$$

$$= \int_0^3 \int_0^{3-y} (2z + 2yz) \bigg|_0^{6-2x-2y} \, dx \, dy$$

$$= \int_0^3 \int_0^{3-y} (12 - 4x + 8y - 4xy - 4y^2) \, dx \, dy$$

$$= \int_0^3 \left[ 12x - 2x^2 + 8xy - 2x^2y - 4xy^2 \right]_0^{3-y} \, dy$$

$$= \int_0^3 (18 + 6y - 10y^2 + 2y^3) \, dy$$

$$= \left[ 18y + 3y^2 - \frac{10y^3}{3} + \frac{y^4}{2} \right]_0^3$$

$$= \frac{63}{2}.$$

Try It Exploration A

TECHNOLOGY If you have access to a computer algebra system that can evaluate triple-iterated integrals, use it to verify the result in Example 1. When you are using such a utility, note that the first step is to convert the triple integral to an iterated integral—this step must be done by hand. To give yourself some practice with this important step, find the limits of integration for the following iterated integrals. Then use a computer to verify that the value is the same as that obtained in Example 1.

$$\int_0^3 \int_0^{3-y} \int_0^{6-2x-2y} (2 + 2y) \, dz \, dx \, dy, \quad \int_0^3 \int_0^y \int_0^{6-2x-2y} (2 + 2y) \, dx \, dy \, dz$$
**EXAMPLE 2 Verifying the Divergence Theorem**

Let $Q$ be the solid region between the paraboloid

$$z = 4 - x^2 - y^2$$

and the $xy$-plane. Verify the Divergence Theorem for

$$\mathbf{F}(x, y, z) = 2\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}.$$  

**Solution** From Figure 15.57 you can see that the outward normal vector for the surface $S_1$ is $\mathbf{N}_1 = -\mathbf{k}$, whereas the outward normal vector for the surface $S_2$ is

$$\mathbf{N}_2 = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}.$$  

So, by Theorem 15.11, you have

\[
\int_S \mathbf{F} \cdot \mathbf{N} \, dS = \int_{S_1} \mathbf{F} \cdot \mathbf{N}_1 \, dS + \int_{S_2} \mathbf{F} \cdot \mathbf{N}_2 \, dS
\]

\[
= \int_{S_1} \mathbf{F} \cdot (-\mathbf{k}) \, dS + \int_{S_2} \mathbf{F} \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \, dS
\]

\[
= \int_{R} -y^2 \, dA + \int_{R} (4xz + 2xy + y^2) \, dA
\]

\[
= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4xz + 2xy + y^2) \, dx \, dy
\]

\[
= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4x(4 - x^2 - y^2) + 2xy) \, dx \, dy
\]

\[
= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (16x - 4x^3 - 4xy^2 + 2xy) \, dx \, dy
\]

\[
= \int_{-2}^{2} \left[ 8x^2 - x^4 - 2x^2y^2 + x^2y \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \, dy
\]

\[
= \int_{-2}^{2} 0 \, dy
\]

\[
= 0.
\]

On the other hand, because

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}[2z] + \frac{\partial}{\partial y}[x] + \frac{\partial}{\partial z}[y^2] = 0 + 0 + 0 = 0$$

you can apply the Divergence Theorem to obtain the equivalent result

\[
\int_Q \text{div } \mathbf{F} \, dV = \int_Q \int_0 \, dV = 0.
\]
EXAMPLE 3  Using the Divergence Theorem

Let $Q$ be the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the $xy$-plane, as shown in Figure 15.58. Find

$$\int_S \mathbf{F} \cdot \mathbf{N} \, dS$$

where $S$ is the surface of $Q$ and

$$\mathbf{F}(x, y, z) = (x^2 + \sin z)\mathbf{i} + (xy + \cos z)\mathbf{j} + e^z\mathbf{k}.$$ 

Solution  Direct evaluation of this surface integral would be difficult. However, by the Divergence Theorem, you can evaluate the integral as follows.

$$\int_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q \text{div} \mathbf{F} \, dV$$

$$= \iiint_Q (2x + x + 0) \, dV$$

$$= \iiint_Q 3x \, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_0^{6-r\cos\theta} (3r \cos \theta) r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (18r^2 \cos \theta - 3r^3 \cos^2 \theta) \, dr \, d\theta$$

$$= \int_0^{2\pi} (48 \cos \theta - 12 \cos^2 \theta) \, d\theta$$

$$= \left[ 48 \sin \theta - 6 \left( \theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi}$$

$$= -12\pi$$

Notice that cylindrical coordinates with $x = r \cos \theta$ and $dV = r \, dz \, dr \, d\theta$ were used to evaluate the triple integral.

Try It  Exploration A

Even though the Divergence Theorem was stated for a simple solid region $Q$ bounded by a closed surface, the theorem is also valid for regions that are the finite unions of simple solid regions. For example, let $Q$ be the solid bounded by the closed surfaces $S_1$ and $S_2$, as shown in Figure 15.59. To apply the Divergence Theorem to this solid, let $S = S_1 \cup S_2$. The normal vector $\mathbf{N}$ to $S$ is given by $-\mathbf{N}_1$ on $S_1$ and by $\mathbf{N}_2$ on $S_2$. So, you can write

$$\int_Q \text{div} \mathbf{F} \, dV = \iiint_Q \mathbf{F} \cdot \mathbf{N} \, dS$$

$$= \int_{S_1} \int F \cdot (-\mathbf{N}_1) \, dS + \int_{S_2} \int F \cdot \mathbf{N}_2 \, dS$$

$$= -\int_{S_1} \int \mathbf{F} \cdot \mathbf{N}_1 \, dS + \int_{S_2} \int \mathbf{F} \cdot \mathbf{N}_2 \, dS.$$
Flux and the Divergence Theorem

To help understand the Divergence Theorem, consider the two sides of the equation

$$\iiint_Q \mathbf{F} \cdot 
abla dV = \iiint_Q \text{div} \mathbf{F} dV.$$  

You know from Section 15.6 that the flux integral on the left determines the total fluid flow across the surface $S$ per unit of time. This can be approximated by summing the fluid flow across small patches of the surface. The triple integral on the right measures this same fluid flow across $S$, but from a very different perspective—namely, by calculating the flow of fluid into (or out of) small cubes of volume $\Delta V_i$. The flux of the $i$th cube is approximately

$$\text{Flux of } i\text{th cube} = \text{div} \mathbf{F}(x_i, y_i, z_i) \Delta V_i$$

for some point $(x_i, y_i, z_i)$ in the $i$th cube. Note that for a cube in the interior of $Q$, the gain (or loss) of fluid through any one of its six sides is offset by a corresponding loss (or gain) through one of the sides of an adjacent cube. After summing over all the cubes in $Q$, the only fluid flow that is not canceled by adjoining cubes is that on the outside edges of the cubes on the boundary. So, the sum

$$\sum_{i=1}^n \text{div} \mathbf{F}(x_i, y_i, z_i) \Delta V_i$$

approximates the total flux into (or out of) $Q$, and therefore through the surface $S$.

To see what is meant by the divergence of $\mathbf{F}$ at a point, consider $\Delta V_\alpha$ to be the volume of a small sphere $S_\alpha$ of radius $\alpha$ and center $(x_0, y_0, z_0)$, contained in region $Q$, as shown in Figure 15.60. Applying the Divergence Theorem to $S_\alpha$ produces

$$\text{Flux of } \mathbf{F} \text{ across } S_\alpha = \iiint_{Q_\alpha} \text{div} \mathbf{F} dV$$

$$= \text{div} \mathbf{F}(x_0, y_0, z_0) \Delta V_\alpha$$

where $Q_\alpha$ is the interior of $S_\alpha$. Consequently, you have

$$\text{div} \mathbf{F}(x_0, y_0, z_0) = \frac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{\Delta V_\alpha}$$

and, by taking the limit as $\alpha \to 0$, you obtain the divergence of $\mathbf{F}$ at the point $(x_0, y_0, z_0)$.

$$\text{div} \mathbf{F}(x_0, y_0, z_0) = \lim_{\alpha \to 0} \frac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{\Delta V_\alpha}$$

= flux per unit volume at $(x_0, y_0, z_0)$

The point $(x_0, y_0, z_0)$ in a vector field is classified as a source, a sink, or incompressible, as follows.

1. **Source**, if $\text{div} \mathbf{F} > 0$ See Figure 15.61(a).
2. **Sink**, if $\text{div} \mathbf{F} < 0$ See Figure 15.61(b).
3. **Incompressible**, if $\text{div} \mathbf{F} = 0$ See Figure 15.61(c).

**NOTE** In hydrodynamics, a source is a point at which additional fluid is considered as being introduced to the region occupied by the fluid. A sink is a point at which fluid is considered as being removed.
EXAMPLE 4 Calculating Flux by the Divergence Theorem

Let $Q$ be the region bounded by the sphere $x^2 + y^2 + z^2 = 4$. Find the outward flux of the vector field $\mathbf{F}(x, y, z) = 2x^3\mathbf{i} + 2y^3\mathbf{j} + 2z^3\mathbf{k}$ through the sphere.

Solution By the Divergence Theorem, you have

\[
\text{Flux across } S = \int_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q \text{div } \mathbf{F} \, dV
\]

\[
= \iiint_Q 6(x^2 + y^2 + z^2) \, dV
\]

\[
= 6 \int_0^{2\pi} \int_0^\pi \int_0^2 \rho^4 \sin \phi \, d\phi \, d\theta \, d\rho
\]

Spherical coordinates

\[
= 6 \int_0^{2\pi} \int_0^\pi 2\pi \rho^4 \sin \phi \, d\phi \, d\rho
\]

\[
= 12\pi \int_0^2 2\rho^4 \, d\rho
\]

\[
= 24\pi \left( \frac{32}{5} \right)
\]

\[
= \frac{768\pi}{5}
\]
Exercises for Section 15.7

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on to view the complete solution of the exercise.

Click on to print an enlarged copy of the graph.

In Exercises 1–4, verify the Divergence Theorem by evaluating

\[ \int_S \mathbf{F} \cdot \mathbf{N} \, dS \]

as a surface integral and as a triple integral.

1. \( \mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k} \)
   
   \( S \): cube bounded by the planes \( x = 0, x = a, y = 0, y = a, \\ z = 0, z = a \)

2. \( \mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k} \)
   
   \( S \): cylinder \( x^2 + y^2 = 4, \ 0 \leq z \leq h \)

3. \( \mathbf{F}(x, y, z) = (2x - y)\mathbf{i} - (2y - z)\mathbf{j} + z\mathbf{k} \)
   
   \( S \): surface bounded by the plane \( 2x + 4y + 2z = 12 \) and the coordinate planes

4. \( \mathbf{F}(x, y, z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k} \)
   
   \( S \): surface bounded by the planes \( y = 4 \) and \( z = 4 - x \) and the coordinate planes

Figure for 1

Figure for 2

Figure for 3

Figure for 4
In Exercises 5–16, use the Divergence Theorem to evaluate
\[ \int_S F \cdot N \, dS \]
and find the outward flux of \( F \) through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results.

5. \( F(x, y, z) = x^2i + y^2j + z^2k \)
   \( S: x = 0, y = 0, z = 0, z = a \)
6. \( F(x, y, z) = x^2z^2i - 2yj + 3xyzk \)
   \( S: x = 0, y = 0, z = 0, z = a \)
7. \( F(x, y, z) = x^2i - 2xyj + xyzz^2k \)
   \( S: z = \sqrt[3]{x^2 - y^2}, z = 0 \)
8. \( F(x, y, z) = xyi + yzj - yzk \)
   \( S: z = \sqrt[3]{x^2 - y^2}, z = 0 \)
9. \( F(x, y, z) = xi + yj + zk \)
   \( S: x^2 + y^2 + z^2 = 4 \)
10. \( F(x, y, z) = xyi + y^2j \)
    \( S: x^2 + y^2 = 9, z = 0, z = 4 \)
11. \( F(x, y, z) = xi + y^2j - zk \)
    \( S: x^2 + y^2 = 9, z = 0, z = 4 \)
12. \( F(x, y, z) = (xy^2 + \cos z)i + (x^2 + \sin z)j + e^z k \)
    \( S: z = \sqrt[3]{x^2 + y^2}, z = 8 \)
13. \( F(x, y, z) = x^2i + yzj + xe^zk \)
    \( S: z = 4 + y, z = 0, x = 0, x = 6, y = 0 \)
14. \( F(x, y, z) = xe^i + ye^j + e^zk \)
    \( S: z = 4 - y, z = 0, x = 0, x = 6, y = 0 \)
15. \( F(x, y, z) = xyi + 4yj + xzk \)
    \( S: x^2 + y^2 + z^2 = 9 \)
16. \( F(x, y, z) = 2(xi + yj + zk) \)
    \( S: z = \sqrt[3]{4 - x^2 - y^2}, z = 0 \)

In Exercises 17 and 18, evaluate
\[ \int_Q \text{curl} F \cdot N \, dS \]
where \( Q \) is the closed surface of the solid bounded by the graphs of \( x = 4 \), \( y = 4 \), and \( z = 9 - y^2 \), and the coordinate planes.

17. \( F(x, y, z) = (4xy + z^2)i + (2x^2 + 6yz)j + 2xzk \)
18. \( F(x, y, z) = xy \cos z i + yz \sin x j + xyzk \)

21. Use the Divergence Theorem to verify that the volume of the solid bounded by a surface \( S \) is
   \[ \int_S x \, dy \, dz = \int_S y \, dz \, dx = \int_S z \, dx \, dy. \]
22. Verify the result of Exercise 21 for the cube bounded by \( x = 0 \)
    \( x = a, y = 0, \) \( y = a, z = 0, a, \) and \( z = a. \)
23. Verify that
   \[ \int_S \text{curl} F \cdot N \, dS = 0 \]
   for any closed surface \( S \).

24. For the constant vector field given by
    \( F(x, y, z) = a_1i + a_2j + a_3k \)
    verify that
   \[ \int_S F \cdot N \, dS = 0 \]
   where \( V \) is the volume of the solid bounded by the closed surface \( S \).

25. Given the vector field
    \( F(x, y, z) = xi + yj + zk \)
    verify that
   \[ \int_S F \cdot N \, dS = 3V \]
   where \( V \) is the volume of the solid bounded by the closed surface \( S \).

26. Given the vector field
    \( F(x, y, z) = xi + yj + zk \)
    verify that
   \[ \frac{1}{|F|} \int_Q F \cdot N \, dS = \frac{3}{|F|} \int_Q dV. \]

In Exercises 27 and 28, prove the identity, assuming that \( Q, S \) and \( N \) meet the conditions of the Divergence Theorem and that the required partial derivatives of the scalar functions \( f \) and \( g \) are continuous. The expressions \( D_N f \) and \( D_N g \) are the derivatives in the direction of the vector \( N \) and are defined by
\[ D_N f = \nabla f \cdot N, \quad D_N g = \nabla g \cdot N. \]

27. \[ \int_Q \left( f \frac{\nabla g}{g} + \frac{\nabla f}{f} \cdot \nabla g \right) \, dV = \int_S (fD_N g) \, dS \]
   \( \text{[Hint: Use div} (fG) = f \text{ div} G + \nabla f \cdot G] \)
28. \[ \int_Q \left( f \frac{\nabla g}{g} - g \frac{\nabla f}{f} \right) \, dV = \int_S (fD_N g - gD_N f) \, dS \]
   \( \text{[Hint: Use Exercise 27 twice.]} \)
Stokes’s Theorem

A second higher-dimension analog of Green’s Theorem is called Stokes’s Theorem, after the English mathematical physicist George Gabriel Stokes. Stokes was part of a group of English mathematical physicists referred to as the Cambridge School, which included William Thomson (Lord Kelvin) and James Clerk Maxwell. In addition to making contributions to physics, Stokes worked with infinite series and differential equations, as well as with the integration results presented in this section.

Stokes’s Theorem gives the relationship between a surface integral over an oriented surface and a line integral along a closed space curve forming the boundary of the surface, as shown in Figure 15.62. The positive direction along the curve is counterclockwise relative to the normal vector of the surface. That is, if you imagine grasping the normal vector with your right hand, with your thumb pointing in the direction of your fingers, your fingers will point in the positive direction, as shown in Figure 15.63.

**THEOREM 15.13  Stokes’s Theorem**

Let $S$ be an oriented surface with unit normal vector $\mathbf{N}$, bounded by a piecewise smooth simple closed curve $C$ with a positive orientation. If $\mathbf{F}$ is a vector field whose component functions have continuous partial derivatives on an open region containing $S$ and $C$, then

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS.
$$

NOTE  The line integral may be written in the differential form $\int_C M\,dx + N\,dy + P\,dz$ or in the vector form $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$. 

---

**George Gabriel Stokes (1819–1903)**

Stokes became a Lucasian professor of mathematics at Cambridge in 1849. Five years later, he published the theorem that bears his name as a prize examination question there.
**EXAMPLE 1** Using Stokes's Theorem

Let \( C \) be the oriented triangle lying in the plane \( 2x + 2y + z = 6 \), as shown in Figure 15.64. Evaluate

\[
\int_C \mathbf{F} \cdot d\mathbf{r}
\]

where \( \mathbf{F}(x, y, z) = -y^2 \mathbf{i} + z \mathbf{j} + x \mathbf{k} \).

**Solution** Using Stokes’s Theorem, begin by finding the curl of \( \mathbf{F} \).

\[
\text{curl } \mathbf{F} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
- y^2 & z & x
\end{vmatrix}
= -\mathbf{i} - 2y \mathbf{k}
\]

Considering \( z = 6 - 2x - 2y = g(x, y) \), you can use Theorem 15.11 for an upward normal vector to obtain

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS
\]

\[
= \int_0^3 \int_0^{3-x} (-i - 2y k) \cdot [-g_x(x, y)i - g_y(x, y)j + k] \, dA
\]

\[
= \int_0^3 \int_0^{3-x} (-i - 2y k) \cdot (2i + 2j + k) \, dA
\]

\[
= \int_0^3 \int_0^{3-x} (2y - 4) \, dx \, dy
\]

\[
= \int_0^3 (-2y^2 + 10y - 12) \, dy
\]

\[
= \left[ -\frac{2y^3}{3} + 5y^2 - 12y \right]_0^3
\]

\[
= -9.
\]

**Try It**

Try evaluating the line integral in Example 1 directly, without using Stokes’s Theorem. One way to do this would be to consider \( C \) as the union of \( C_1 \), \( C_2 \), and \( C_3 \), as follows.

\( C_1: \mathbf{r}_1(t) = (3 - t) \mathbf{i} + t \mathbf{j}, \quad 0 \leq t \leq 3 \)

\( C_2: \mathbf{r}_2(t) = (6 - t) \mathbf{j} + (2t - 6) \mathbf{k}, \quad 3 \leq t \leq 6 \)

\( C_3: \mathbf{r}_3(t) = (t - 6) \mathbf{i} + (18 - 2t) \mathbf{k}, \quad 6 \leq t \leq 9 \)

The value of the line integral is

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{r}_1'(t) \, dt + \int_{C_2} \mathbf{F} \cdot \mathbf{r}_2'(t) \, dt + \int_{C_3} \mathbf{F} \cdot \mathbf{r}_3'(t) \, dt
\]

\[
= \int_0^3 t^2 \, dt + \int_3^6 (-2t + 6) \, dt + \int_6^9 (-2t + 12) \, dt
\]

\[
= 9 - 9 - 9
\]

\[
= -9.
\]
**EXAMPLE 2 Verifying Stokes’s Theorem**

Verify Stokes’s Theorem for \( \mathbf{F}(x, y, z) = 2z \mathbf{i} + x \mathbf{j} + y^2 \mathbf{k} \), where \( S \) is the surface of the paraboloid \( z = 4 - x^2 - y^2 \) and \( C \) is the trace of \( S \) in the \( xy \)-plane, as shown in Figure 15.65.

**Solution** As a surface integral, you have \( z = g(x, y) = 4 - x^2 - y^2 \) and

\[
\mathbf{curl} \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2z & x & y^2
\end{vmatrix} = 2y \mathbf{i} + 2 \mathbf{j} + \mathbf{k}.
\]

By Theorem 15.11 for an upward normal vector \( \mathbf{N} \), you obtain

\[
\int_S (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = \int_R (2y \mathbf{i} + 2 \mathbf{j} + \mathbf{k}) \cdot (2x \mathbf{i} + 2y \mathbf{j} + \mathbf{k}) \, dA
\]

\[
= \int_{-2}^{2} \int_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4xy + 4y + 1) \, dx \, dy
\]

\[
= \int_{-2}^{2} \left[ 2x^2y + (4y + 1)x \right]_{\sqrt{4-y^2}}^{\sqrt{4-y^2}} \, dy
\]

\[
= \int_{-2}^{2} 2(4y + 1) \sqrt{4 - y^2} \, dy
\]

\[
= \int_{-2}^{2} (8y \sqrt{4 - y^2} + 2 \sqrt{4 - y^2}) \, dy
\]

\[
= \left[ -\frac{8}{3} (4 - y^2)^{3/2} + y \sqrt{4 - y^2} + 4 \arcsin \frac{y}{2} \right]_{-2}^{2}
\]

\[
= 4\pi.
\]

As a line integral, you can parametrize \( C \) by

\[
\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + 0 \mathbf{k}, \quad 0 \leq t \leq 2\pi.
\]

For \( \mathbf{F}(x, y, z) = 2z \mathbf{i} + x \mathbf{j} + y^2 \mathbf{k} \), you obtain

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \, dx + N \, dy + P \, dz
\]

\[
= \int_{C} 2z \, dx + x \, dy + y^2 \, dz
\]

\[
= \int_{0}^{2\pi} (0 + 2 \cos t(2 \cos t) + 0) \, dt
\]

\[
= \int_{0}^{2\pi} 4 \cos^2 t \, dt
\]

\[
= 2 \int_{0}^{2\pi} (1 + \cos 2t) \, dt
\]

\[
= 2 \left[ t + \frac{1}{2} \sin 2t \right]_{0}^{2\pi}
\]

\[
= 4\pi.
\]
Physical Interpretation of Curl

Stokes’s Theorem provides insight into a physical interpretation of curl. In a vector field \( \mathbf{F} \), let \( S_\alpha \) be a small circular disk of radius \( \alpha \), centered at \((x, y, z)\) and with boundary \( C_\alpha \), as shown in Figure 15.66. At each point on the circle \( C_\alpha \), \( \mathbf{F} \) has a normal component \( \mathbf{F} \cdot \mathbf{N} \) and a tangential component \( \mathbf{F} \cdot \mathbf{T} \). The more closely \( \mathbf{F} \) and \( \mathbf{T} \) are aligned, the greater the value of \( \mathbf{F} \cdot \mathbf{T} \). So, a fluid tends to move along the circle rather than across it. Consequently, you say that the line integral around \( C_\alpha \) measures the circulation of \( \mathbf{F} \) around \( C_\alpha \). That is,

\[
\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T} \, ds = \text{circulation of } \mathbf{F} \text{ around } C_\alpha.
\]

Now consider a small disk \( S_\alpha \) to be centered at some point \((x, y, z)\) on the surface \( S \), as shown in Figure 15.67. On such a small disk, \( \text{curl } \mathbf{F} \) is nearly constant, because it varies little from its value at \((x, y, z)\). Moreover, \( \text{curl } \mathbf{F} \cdot \mathbf{N} \) is also nearly constant on \( S_\alpha \), because all unit normals to \( S_\alpha \) are about the same. Consequently, Stokes’s Theorem yields

\[
\int_{S_\alpha} \text{curl } \mathbf{F} \cdot \mathbf{N} \, dS = \left( \text{curl } \mathbf{F} \right) \cdot \mathbf{N} \int_{S_\alpha} dS
\]

So,

\[
\left( \text{curl } \mathbf{F} \right) \cdot \mathbf{N} = \frac{\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T} \, ds}{\pi \alpha^2} = \frac{\text{circulation of } \mathbf{F} \text{ around } C_\alpha}{\text{area of disk } S_\alpha} = \text{rate of circulation}.
\]

Assuming conditions are such that the approximation improves for smaller and smaller disks \((\alpha \to 0)\), it follows that

\[
\left( \text{curl } \mathbf{F} \right) \cdot \mathbf{N} = \lim_{\alpha \to 0} \frac{1}{\pi \alpha^2} \int_{C_\alpha} \mathbf{F} \cdot \mathbf{T} \, ds
\]

which is referred to as the rotation of \( \mathbf{F} \) about \( \mathbf{N} \). That is,

\[
\text{curl } \mathbf{F}(x, y, z) \cdot \mathbf{N} = \text{rotation of } \mathbf{F} \text{ about } \mathbf{N} \text{ at } (x, y, z).
\]

In this case, the rotation of \( \mathbf{F} \) is maximum when \( \text{curl } \mathbf{F} \) and \( \mathbf{N} \) have the same direction. Normally, this tendency to rotate will vary from point to point on the surface \( S \), and Stokes’s Theorem

\[
\int_S \left( \text{curl } \mathbf{F} \right) \cdot \mathbf{N} \, dS = \int_C \mathbf{F} \cdot d\mathbf{r}
\]

says that the collective measure of this rotational tendency taken over the entire surface \( S \) (surface integral) is equal to the tendency of a fluid to circulate around the boundary \( C \) (line integral).
EXAMPLE 3  An Application of Curl

A liquid is swirling around in a cylindrical container of radius 2, so that its motion is described by the velocity field

\[ \mathbf{F}(x, y, z) = -y\sqrt{x^2 + y^2}\mathbf{i} + x\sqrt{x^2 + y^2}\mathbf{j} \]

as shown in Figure 15.68. Find

\[ \int_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS \]

where \( S \) is the upper surface of the cylindrical container.

**Solution**  The curl of \( \mathbf{F} \) is given by

\[
\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y\sqrt{x^2 + y^2} & x\sqrt{x^2 + y^2} & 0 \end{vmatrix} = 3\sqrt{x^2 + y^2}\mathbf{k}.
\]

Letting \( \mathbf{N} = \mathbf{k} \), you have

\[
\int_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \int_R \int_{\theta=0}^{2\pi} \int_{r=0}^{2\pi} 3\sqrt{x^2 + y^2} \, dr \, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{2\pi} 8 \, d\theta = 16\pi.
\]

**Try It**  Exploration A

*NOTE:* If \( \text{curl } \mathbf{F} = \mathbf{0} \) throughout region \( Q \), the rotation of \( \mathbf{F} \) about each unit normal \( \mathbf{N} \) is 0. That is, \( \mathbf{F} \) is irrotational. From earlier work, you know that this is a characteristic of conservative vector fields.

**Summary of Integration Formulas**

*Fundamental Theorem of Calculus:*

\[ \int_a^b F'(x) \, dx = F(b) - F(a) \]

*Fundamental Theorem of Line Integrals:*

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a)) \]

*Green’s Theorem:*

\[
\int_C M \, dx + N \, dy = \int_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_R \left( \text{curl } \mathbf{F} \right) \cdot \mathbf{k} \, dA
\]

\[
\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \int_R \int \text{div } \mathbf{F} \, dA
\]

*Divergence Theorem:*

\[
\int_S \int \mathbf{F} \cdot \mathbf{N} \, dS = \int_Q \int \text{div } \mathbf{F} \, dV
\]

*Stokes’s Theorem:*

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS \]
Exercises for Section 15.8

The symbol \( \equiv \) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on [S] to view the complete solution of the exercise.

Click on [M] to print an enlarged copy of the graph.

In Exercises 1–6, find the curl of the vector field \( \mathbf{F} \).
1. \( \mathbf{F}(x, y, z) = (2y - z)i + xyj + e^z k \)
2. \( \mathbf{F}(x, y, z) = x^2i + y^2j + z^2 k \)
3. \( \mathbf{F}(x, y, z) = 2zi - 4x^2j + \arctan xk \)
4. \( \mathbf{F}(x, y, z) = x \sin yj - y \cos xj + y^2z k \)
5. \( \mathbf{F}(x, y, z) = e^{x^2 + y^2}i + e^{x^2 + y^2}j + xyzk \)
6. \( \mathbf{F}(x, y, z) = \arcsin yi + \sqrt{1 - x^2}j + y^2k \)

In Exercises 7–10, verify Stokes’s Theorem by evaluating
\[
\int_C \mathbf{F} \cdot T \, ds = \oint_C \mathbf{F} \cdot d\mathbf{r}
\]
as a line integral and as a double integral.

7. \( \mathbf{F}(x, y, z) = (-y + z)i + (x - z)j + (x - y)k \)
\[S: z = \sqrt{1 - x^2 - y^2} \]
8. \( \mathbf{F}(x, y, z) = (-y + z)i + (x - z)j + (x - y)k \)
\[S: z = 4 - x^2 - y^2, \quad z \geq 0 \]
9. \( \mathbf{F}(x, y, z) = xyzi + y^2j + zk \)
\[S: 3x + 4y + 2z = 12, \quad \text{first octant} \]
10. \( \mathbf{F}(x, y, z) = z^2i + x^2j + y^2k \)
\[S: z = y^2, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a \]

In Exercises 11–20, use Stokes’s Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \). Use a computer algebra system to verify your results. In each case, \( C \) is oriented counterclockwise as viewed from above.

11. \( \mathbf{F}(x, y, z) = 2yi + 3xj + xk \)
\( C: \) triangle with vertices \((0, 0, 0), (0, 2, 0), (1, 1, 1)\)
12. \( \mathbf{F}(x, y, z) = \arctan \frac{x}{y}i + \ln \sqrt{x^2 + y^2}j + k \)
\( C: \) triangle with vertices \((0, 0, 0), (1, 1, 1), (0, 0, 2)\)
13. \( \mathbf{F}(x, y, z) = z^2i + x^2j + y^2k \)
\[S: z = 4 - x^2 - y^2, \quad z \geq 0 \]
14. \( \mathbf{F}(x, y, z) = 4xyi + y^2j + 4xk \)
\[S: z = 9 - x^2 - y^2, \quad z \geq 0 \]
15. \( \mathbf{F}(x, y, z) = z^2i + y^2j + xzk \)
\[S: z = \sqrt{4 - x^2 - y^2} \]
16. \( \mathbf{F}(x, y, z) = x^2i + z^2j - xyzk \)
\[S: z = \sqrt{4 - x^2 - y^2} \]
17. \( \mathbf{F}(x, y, z) = -\ln \sqrt{x^2 + y^2}i + \arctan \frac{x}{y}j + k \)
\[S: z = 9 - 2x - 3y \text{ over one petal of } r = 2 \sin 2\theta \text{ in the first octant} \]
18. \( \mathbf{F}(x, y, z) = y^2i + (2 - 3y)j + (x^2 + y^2)k, \quad x^2 + y^2 \leq 16 \)
\[S: \text{the first-octant portion of } x^2 + z^2 = 16 \text{ over } x^2 + y^2 = 16 \]
19. \( \mathbf{F}(x, y, z) = xyzi + y^2j + zk \)
\[S: z = x^2, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a \]
\( N \) is the downward unit normal to the surface.

20. \( \mathbf{F}(x, y, z) = xyzi + y^2j + zk, \quad x^2 + y^2 \leq a^2 \)
\( S: \) the first-octant portion of \( z = x^2 \text{ over } x^2 + y^2 = a^2 \)

Motion of a Liquid In Exercises 21 and 22, the motion of a liquid in a cylindrical container of radius 1 is described by the velocity field \( \mathbf{F}(x, y, z) \). Find
\[
\oint_C (\nabla \times \mathbf{F}) \cdot \mathbf{N} \, dS
\]
where \( S \) is the upper surface of the cylindrical container.

21. \( \mathbf{F}(x, y, z) = i + j - 2k \quad 22. \mathbf{F}(x, y, z) = -zi + yk \)

Writing About Concepts

23. State Stokes’s Theorem.
24. Give a physical interpretation of curl.
25. According to Stokes’s Theorem, what can you conclude about the circulation in a field whose curl is 0? Explain your reasoning.

26. Let \( f \) and \( g \) be scalar functions with continuous partial derivatives, and let \( C \) and \( S \) satisfy the conditions of Stokes’s Theorem. Verify each identity.
   (a) \( \oint_C (f \nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{N} \, dS \)
   (b) \( \oint_C (f \nabla f) \cdot d\mathbf{r} = 0 \)
   (c) \( \oint_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0 \)

27. Demonstrate the results of Exercise 26 for the function \( f(x, y, z) = xyz \text{ and } g(x, y, z) = z \). Let \( S \) be the hemispher \( z = \sqrt{4 - x^2 - y^2} \).

28. Let \( C \) be a constant vector. Let \( S \) be an oriented surface with \( 4 \) unit normal vector \( \mathbf{N} \), bounded by a smooth curve \( C \). Prove that
\[
\oint_C \mathbf{N} \cdot d\mathbf{S} = \frac{1}{2} \iint_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r}
\]

Putnam Exam Challenge

29. Let \( \mathbf{G}(x, y) = \begin{pmatrix} -y/(x^2 + 4y^2) & x/(x^2 + 4y^2) & 0 \end{pmatrix} \).

Prove or disprove that there is a vector-valued function \( \mathbf{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z)) \) with the following properties.
   (i) \( M, N, P \) have continuous partial derivatives for all \((x, y, z) \neq (0, 0, 0)\);
   (ii) \( \text{Curl} \mathbf{F} = 0 \) for all \((x, y, z) \neq (0, 0, 0)\);
   (iii) \( \mathbf{F}(x, y, 0) = \mathbf{G}(x, y) \).

This problem was composed by the Committee on the Putnam Prize Competition.
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The symbol \( \mathbb{C} \) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on \( \text{S} \) to view the complete solution of the exercise.

Click on \( \text{M} \) to print an enlarged copy of the graph.

In Exercises 1 and 2, sketch several representative vectors in the vector field. Use a computer algebra system to verify your results.

1. \( \mathbf{F}(x, y, z) = xi + j + 2k \)

2. \( \mathbf{F}(x, y) = i - 2yj \)

In Exercises 3 and 4, find the gradient vector field for the scalar function.

3. \( f(x, y, z) = 8x^2 + xy + z^2 \)

4. \( f(x, y, z) = x^2e^z \)

In Exercises 5–12, determine if the vector field is conservative. If it is, find a potential function for the vector field.

5. \( \mathbf{F}(x, y) = \frac{1}{y}i - \frac{y}{x^2}j \)

6. \( \mathbf{F}(x, y) = -\frac{y}{x^2}i + \frac{1}{x}j \)

7. \( \mathbf{F}(x, y) = (6xy^2 - 3z^2)i + (6x^2y + 3y^2 - 7)j \)

8. \( \mathbf{F}(x, y, z) = -(2y^3 \sin 2x)i + 3y^2(1 + \cos 2x)j \)

9. \( \mathbf{F}(x, y, z) = (4xy + z)j + (2x^2 + 6y)j + 2z k \)

10. \( \mathbf{F}(x, y, z) = (4xy + z^2)i + (2x^2 + 6yz)j + 2xz k \)

11. \( \mathbf{F}(x, y, z) = \frac{yz i - xz j - xy k}{y^2z^2} \)

12. \( \mathbf{F}(x, y, z) = \sin z(yi + xj + k) \)

In Exercises 13–20, find (a) the divergence of the vector field \( \mathbf{F} \) and (b) the curl of the vector field \( \mathbf{F} \).

13. \( \mathbf{F}(x, y, z) = x^2i + y^2j + z^2 k \)

14. \( \mathbf{F}(x, y, z) = xy^2j - zx^2 k \)

15. \( \mathbf{F}(x, y, z) = (\cos y + y \cos x)i + (\sin x - x \sin y)j + xyz k \)

16. \( \mathbf{F}(x, y, z) = (3x - y)i + (y - 2)j + (z - 3x) k \)

17. \( \mathbf{F}(x, y, z) = \arcsin xi + xy^2 j + yz^2 k \)

18. \( \mathbf{F}(x, y, z) = (x^2 - y)i - (x + \sin^2 y)j \)

19. \( \mathbf{F}(x, y, z) = \ln(x^2 + y^2) i + \ln(x^2 + y^2) j + zk \)

20. \( \mathbf{F}(x, y, z) = \frac{x}{x} i + \frac{y}{y} j + \frac{z}{z} k \)

In Exercises 21–26, evaluate the line integral along the given path(s).

21. \( \int_C (x^2 + y^2)ds \)
   (a) \( C \): line segment from \((-1, -1)\) to \((2, 2)\)
   (b) \( C \): \( x^2 + y^2 = 16 \), one revolution counterclockwise, starting at \((4, 0)\)

22. \( \int_C xy ds \)
   (a) \( C \): line segment from \((0, 0)\) to \((5, 4)\)
   (b) \( C \): counterclockwise around the triangle with vertices \((0, 0), (4, 0), (0, 2)\)

23. \( \int_C (x^2 + y^2)ds \)
   \( C \): \( \mathbf{r}(t) = (\cos t + t \sin t)i + (\sin t - t \cos t)j \), \( 0 \leq t \leq 2\pi \)

24. \( \int_C xds \)
   \( C \): \( \mathbf{r}(t) = (t - \sin t)i + (1 - \cos t)j \), \( 0 \leq t \leq 2\pi \)

25. \( \int_C (2x - y)dx + (x + 3y)dy \)
   (a) \( C \): line segment from \((0, 0)\) to \((2, -3)\)
   (b) \( C \): counterclockwise around the circle \( x = 3 \cos t \)
   \( y = 3 \sin t \)

26. \( \int_C (2x - y)dx + (x + 3y)dy \)
   \( C \): \( \mathbf{r}(t) = (\cos t + t \sin t)i + (\sin t - t \sin t)j \), \( 0 \leq t \leq \pi/2 \)

In Exercises 27 and 28, use a computer algebra system to evaluate the line integral over the given path.

27. \( \int_C (2x + y)ds \)

28. \( \int_C (x^2 + y^2 + z^2)ds \)

\( r(t) = a \cos^3 t i + a \sin^3 t j \), \( r(t) = ti + tj + t^{3/2}k \), \( 0 \leq t \leq \pi/2 \)

\( 0 \leq t \leq 4 \)

Lateral Surface Area In Exercises 29 and 30, find the lateral surface area over the curve \( C \) in the \( xy \)-plane and under the surface \( z = f(x, y) \).

29. \( f(x, y) = 5 + \sin(x + y) \)
   \( C \): \( y = 3x \) from \((0, 0)\) to \((2, 6)\)

30. \( f(x, y) = 12 - x - y \)
   \( C \): \( y = x^2 \) from \((0, 0)\) to \((2, 4)\)

In Exercises 31–36, evaluate \( \int_C \mathbf{F} \cdot dr \).

31. \( \mathbf{F}(x, y) = xyi + x^2j \)
   \( C \): \( \mathbf{r}(t) = t^2i + t^3j \), \( 0 \leq t \leq 1 \)

32. \( \mathbf{F}(x, y) = (x - y)i + (x + y)j \)
   \( C \): \( \mathbf{r}(t) = 4 \cos t i + 3 \sin t j \), \( 0 \leq t \leq 2\pi \)

33. \( \mathbf{F}(x, y, z) = xi + yj + zk \)
   \( C \): \( \mathbf{r}(t) = 2 \cos t i + 2 \sin t j + tk \), \( 0 \leq t \leq 2\pi \)

34. \( \mathbf{F}(x, y, z) = (2y - z)i + (z - x)j + (x - y)k \)
   \( C \): curve of intersection of \( x^2 + z^2 = 4 \) and \( y^2 + z^2 = 4 \) from \((2, 2, 0)\) to \((0, 0, 2)\)

35. \( \mathbf{F}(x, y, z) = (y - z)i + (z - x)j + (x - y)k \)
   \( C \): curve of intersection of \( z = x^2 + y^2 \) and \( x + y = 0 \) from \((-2, -2, 8)\) to \((2, -2, 8)\)

36. \( \mathbf{F}(x, y, z) = (x^2 - z)i + (y^2 + z)j + xk \)
   \( C \): curve of intersection of \( z = x^2 \) and \( x^2 + y^2 = 4 \) from \((0, -2, 0)\) to \((2, 0, 0)\)
In Exercises 37 and 38, use a computer algebra system to evaluate the line integral.

37. \( \int_c (x^2 + y^2) \, dx \) 
\( C \): \( y = x^2 \) from \((0, 0)\) to \((2, 4)\) and \( y = 2x \) from \((2, 4)\) to \((0, 0)\)

38. \( \int_c \mathbf{F} \cdot \, d\mathbf{r} \) 
\( \mathbf{F}(x, y) = (2x - y)i + (2y - x)j \) 
\( C \): \( r(t) = (2 \cos t + 2t \sin t)i + (2 \sin t - 2t \cos t)j \), 
\( 0 \leq t \leq \pi \)

39. Work: Find the work done by the force field \( \mathbf{F} = xi - \sqrt{y}j \) along the path \( y = x^{3/2} \) from \((0, 0)\) to \((4, 8)\).

40. Work: A 20-ton aircraft climbs 2000 feet while making a 90° turn in a circular arc of radius 10 miles. Find the work done by the engines.

In Exercises 41 and 42, evaluate the integral using the Fundamental Theorem of Line Integrals.

41. \( \int_c 2xyz \, dx + x^2z \, dy + x^2y \, dz \) 
\( C \): smooth curve from \((0, 0, 0)\) to \((1, 4, 3)\)

42. \( \int_c y \, dx + x \, dy + \frac{1}{z} \, dz \) 
\( C \): smooth curve from \((0, 0, 1)\) to \((4, 4, 4)\)

43. Evaluate the line integral \( \int_c y^2 \, dx + 2xy \, dy \).
   (a) \( C \): \( r(t) = (1 + 3t)i + (1 + t)j \), \( 0 \leq t \leq 1 \)
   (b) \( C \): \( r(t) = ti + \sqrt{t}j \), \( 1 \leq t \leq 4 \)
   (c) Use the Fundamental Theorem of Line Integrals, where \( C \) is a smooth curve from \((1, 1)\) to \((4, 2)\).

44. Area and Centroid: Consider the region bounded by the \( x \)-axis and one arch of the cycloid with parametric equations \( x = a(\theta - \sin \theta) \) and \( y = a(1 - \cos \theta) \). Use line integrals to find (a) the area of the region and (b) the centroid of the region.

In Exercises 45–50, use Green’s Theorem to evaluate the line integral.

45. \( \int_c y \, dx + 2x \, dy \) 
\( C \): boundary of the square with vertices \((0, 0)\), \((0, 2)\), \((2, 0)\), \((2, 2)\)

46. \( \int_c xy \, dx + (x^2 + y^2) \, dy \) 
\( C \): boundary of the square with vertices \((0, 0)\), \((0, 2)\), \((2, 0)\), \((2, 2)\)

47. \( \int_c xy^2 \, dx + x^2y \, dy \) 
\( C \): \( x = 4 \cos t \), \( y = 2 \sin t \)

48. \( \int_c (x^2 - y^2) \, dx + 2xy \, dy \)
\( C \): \( x^2 + y^2 = a^2 \)

49. \( \int_c xy \, dx + x^2 \, dy \)
\( C \): boundary of the region between the graphs of \( y = x^2 \) and \( y = x \)

50. \( \int_c y^2 \, dx + x^{4/3} \, dy \)
\( C \): \( x^{2/3} + y^{2/3} = 1 \)

In Exercises 51 and 52, use a computer algebra system to graph the surface represented by the vector-valued function.

51. \( \mathbf{r}(u, v) = \sec u \cos v \, i + (1 + 2 \tan u) \sin v \, j + 2u \, k \)
\( 0 \leq u \leq \frac{\pi}{3} \), \( 0 \leq v \leq 2\pi \)

52. \( \mathbf{r}(u, v) = e^{-u/4} \cos v \, i + e^{-u/4} \sin v \, j + \frac{u}{6} \, k \)
\( 0 \leq u \leq 4 \), \( 0 \leq v \leq 2\pi \)

53. Investigation: Consider the surface represented by the vector-valued function
\( \mathbf{r}(u, v) = 3 \cos v \cos u \, i + 3 \cos v \sin u \, j + \sin v \, k \).
Use a computer algebra system to do the following.
(a) Graph the surface for \( 0 \leq u \leq 2\pi \) and \( -\frac{\pi}{2} \leq v \leq \frac{\pi}{2} \).
(b) Graph the surface for \( 0 \leq u \leq 2\pi \) and \( \frac{\pi}{4} \leq v \leq \frac{\pi}{2} \).
(c) Graph the surface for \( 0 \leq u \leq \frac{\pi}{4} \) and \( 0 \leq v \leq \frac{\pi}{2} \).
(d) Graph and identify the space curve for \( 0 \leq u \leq 2 \pi \) and
\( v = \frac{\pi}{4} \).
(e) Approximate the area of the surface graphed in part (b).
(f) Approximate the area of the surface graphed in part (c).

54. Evaluate the surface integral \( \int_S z \, dS \) over the surface \( S \):
\( \mathbf{r}(u, v) = (u + v) \, i + (u - v) \, j + \sin v \, k \)
where \( 0 \leq u \leq 2 \) and \( 0 \leq v \leq \pi \).

55. Use a computer algebra system to graph the surface \( S \) and approximate the surface integral
\( \int_S (x + y) \, dS \)
where \( S \) is the surface
\( \mathbf{r}(u, v) = u \cos v \, i + u \sin v \, j + (u - 1)(2 - u) \, k \)
over \( 0 \leq u \leq 2 \) and \( 0 \leq v \leq 2\pi \).
56. **Mass**  
A cone-shaped surface lamina $S$ is given by  
$$z = a(a - \sqrt{x^2 + y^2}), \quad 0 \leq z \leq a^2.$$  
At each point on $S$, the density is proportional to the distance  
between the point and the $z$-axis.
(a) Sketch the cone-shaped surface.
(b) Find the mass $m$ of the lamina.

In Exercises 57 and 58, verify the Divergence Theorem by  
evaluating  
\[ \int_S \mathbf{F} \cdot \mathbf{N} \, dS \]  
as a surface integral and as a triple integral.

57. \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k} \)  
\( Q \): solid region bounded by the coordinate planes and the plane  
\( 2x + 3y + 4z = 12 \)

58. \( \mathbf{F}(x, y, z) = xi + yj + zk \)  
\( Q \): solid region bounded by the coordinate planes and the plane  
\( 2x + 3y + 4z = 12 \)

In Exercises 59 and 60, verify Stokes’s Theorem by evaluating  
\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]  
as a line integral and as a double integral.

59. \( \mathbf{F}(x, y, z) = (\cos y + y \cos x) \mathbf{i} + (\sin x - x \sin y) \mathbf{j} + xyz \mathbf{k} \)  
\( S \): portion of $z = y^2$ over the square in the $xy$-plane with  
vertices $(0, 0), (a, 0), (a, a), (0, a)$  
\( \mathbf{N} \) is the upward unit normal vector to the surface.

60. \( \mathbf{F}(x, y, z) = (x - z) \mathbf{i} + (y - z) \mathbf{j} + x^2 \mathbf{k} \)  
\( S \): first-octant portion of the plane $3x + y + 2z = 12$

61. Prove that it is not possible for a vector field with twice-  
differentiable components to have a curl of \( \mathbf{a}i + \mathbf{b}j + \mathbf{c}k \).
The symbol \( \heartsuit \) indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on [S] to view the complete solution of the exercise.

Click on [M] to print an enlarged copy of the graph.

1. Heat flows from areas of higher temperature to areas of lower temperature in the direction of greatest change. As a result, measuring heat flux involves the gradient of the temperature. The flux depends on the area of the surface. It is the normal direction to the surface that is important, because heat that flows in directions tangential to the surface will give no heat loss. So, assume that the heat flux across a portion of the surface of area \( \Delta S \) is given by \( \Delta H = -k \nabla T \cdot N \, dS \), where \( T \) is the temperature, \( N \) is the unit normal vector to the surface in the direction of the heat flow, and \( k \) is the thermal diffusivity of the material. The heat flux across the surface \( S \) is given by

\[
H = \int_S -k \nabla T \cdot N \, dS.
\]

Consider a single heat source located at the origin with temperature

\[
T(x, y, z) = \frac{25}{\sqrt{x^2 + y^2 + z^2}}.
\]

(a) Calculate the heat flux across the surface

\[
S = \{(x, y, z) : z = \sqrt{1 - x^2}, \quad -\frac{1}{2} \leq x \leq \frac{1}{2}, \quad 0 \leq y \leq 1\}
\]

as shown in the figure.

(b) Repeat the calculation in part (a) using the parametrization

\[
x = \cos u, \quad y = v, \quad z = \sin u, \quad \frac{\pi}{3} \leq u \leq \frac{2\pi}{3}, \quad 0 \leq v \leq 1.
\]

2. Consider a single heat source located at the origin with temperature

\[
T(x, y, z) = \frac{25}{\sqrt{x^2 + y^2 + z^2}}.
\]

(a) Calculate the heat flux across the surface

\[
S = \{(x, y, z) : z = \sqrt{1 - x^2 - y^2}, \quad x^2 + y^2 \leq 1\}
\]

as shown in the figure.

(b) Repeat the calculation in part (a) using the parametrization

\[
x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u, \quad 0 \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq 2\pi.
\]

3. Consider a wire of density \( \rho(x, y, z) \) given by the space curve

\[
C: \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b.
\]

The moments of inertia about the \( x \)-, \( y \)-, and \( z \)-axes are given by

\[
I_x = \int_C (y^2 + z^2)\rho(x, y, z) \, ds \quad I_y = \int_C (x^2 + z^2)\rho(x, y, z) \, ds \quad I_z = \int_C (x^2 + y^2)\rho(x, y, z) \, ds.
\]

Find the moments of inertia for a wire of uniform density \( \rho = 1 \) in the shape of the helix

\[
\mathbf{r}(t) = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j} + \frac{2t}{3}\mathbf{k}, \quad 0 \leq t \leq 2\pi \text{ (see figure)}.
\]

4. Find the moments of inertia for the wire of density \( \rho = \frac{1}{1 + t^2} \) given by the curve

\[
C: \mathbf{r}(t) = \frac{t^2}{2}\mathbf{i} + \mathbf{j} + \frac{2\sqrt{2}t^{3/2}}{3}\mathbf{k}, \quad 0 \leq t \leq 1 \text{ (see figure)}.
\]
5. Use a line integral to find the area bounded by one arch of the cycloid
\[ x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t), \quad 0 \leq t \leq 2\pi \]
as shown in the figure.

6. Use a line integral to find the area bounded by the two loops of the eight curve
\[ x(t) = \frac{1}{2} \sin 2t, \quad y(t) = \sin t, \quad 0 \leq t \leq 2\pi \]
as shown in the figure.

7. The force field \( \mathbf{F}(x, y) = (x + y)\mathbf{i} + (x^2 + 1)\mathbf{j} \) acts on an object moving from the point \((0, 0)\) to the point \((0, 1)\), as shown in the figure.

(a) Find the work done if the object moves along the path \(x = 0, 0 \leq y \leq 1\).
(b) Find the work done if the object moves along the path \(x = y - y^2, 0 \leq y \leq 1\).
(c) Suppose the object moves along the path \(x = c(y - y^2), 0 \leq y \leq 1, c > 0\). Find the value of the constant \(c\) that minimizes the work.

8. The force field \( \mathbf{F}(x, y) = (3x^2y^3)i + (2x^3)j \) is shown in the figure below. Three particles move from the point \((1, 1)\) to the point \((2, 4)\) along different paths. Explain why the work done is the same for each particle, and find the value of the work.

9. Let \( S \) be a smooth oriented surface with normal vector \( \mathbf{N} \) bounded by a smooth simple closed curve \( C \). Let \( \mathbf{v} \) be a constant vector, and prove that
\[ \int_C (2\mathbf{v} \cdot \mathbf{N}) \, dS = \int_C (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{r}. \]

10. How does the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) compare with the magnitude of the work done by the force field
\[ \mathbf{F}(x, y) = -\frac{1}{2}x\mathbf{i} + \frac{1}{2}y\mathbf{j} \]
on a particle that moves once around the ellipse (see figure)?

11. A cross section of Earth’s magnetic field can be represented as a vector field in which the center of Earth is located at the origin and the positive y-axis points in the direction of the magnetic north pole. The equation for this field is
\[ \mathbf{F}(x, y) = M(x, y)i + N(x, y)j \]
\[ = \frac{m}{(x^2 + y^2)^{3/2}} [3xyi + (2y^2 - x^2)j] \]
where \(m\) is the magnetic moment of Earth. Show that this vector field is conservative.