2-5 HW

(21) $\frac{dy}{dx} = \frac{-y}{x}$ $(-4,-1)$: $-\frac{1}{4}$

(37) $\frac{dy}{dx} = \boxed{\frac{x}{y}} = xy^{-1}$

Quot:

$$\frac{d^2y}{dx^2} = \frac{y(1) - x(1)\frac{dy}{dx}}{y^2} = \frac{y - x\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2}$$

$$= \left(\frac{y - \frac{x^2}{y}}{y^2}\right)y = \frac{y^2 - x^2}{y^3}$$

Prod: $\frac{d^2y}{dx^2} = x(-1y^{-2})\boxed{\frac{dy}{dx}} + y^{-1}(1)$

$$= \frac{-x}{y^2}\left(\frac{x}{y}\right) + \frac{1}{y} = \frac{-x^2}{y^3} + \frac{1}{y} = \frac{-x^2+y^2}{y^3}$$

(43) $x^2+y^2=25$ $\qquad \frac{9}{3}+\frac{16}{3}=\frac{25}{3}$

$$\frac{dy}{dx}=\frac{-x}{y}$$

$(4,3)$: $\frac{dy}{dx}=\frac{-4}{3}$

Tangent: $3=-\frac{4}{3}(4)+b$ $\qquad y=-\frac{4}{3}x+\frac{25}{3}$

$\frac{25}{3}=b$

Normal: $3=\frac{3}{\cancel{4}}(\cancel{4})+b$ $\qquad y=\frac{3}{4}x$

$0=b$

$(-3,4)$ $\frac{dy}{dx}=\frac{3}{4}$ $\qquad \frac{16}{4}+\frac{9}{4}=\frac{25}{4}$

Tangent: $4=\frac{3}{4}(-3)+b$ $\qquad y=\frac{3}{4}x+\frac{25}{4}$

$\frac{25}{4}=b$

Normal: $4=-\frac{4}{\cancel{3}}(-\cancel{3})+b$ $\qquad y=-\frac{4}{3}x$

$0=b$

(47) $25x^2+16y^2+200x-160y+400=0$

$$50x+32y\frac{dy}{dx}+200-160\frac{dy}{dx}=0$$

$$(32y-160)\frac{dy}{dx}=-50x-200$$

$$\frac{dy}{dx}=\frac{-50x-200}{32y-160}$$

Horiz: $\frac{0}{1}=\frac{-50x-200}{32y-160}$

$-50x-200=0$

$-50x=200$

$x=-4$

$x=-4$: $25(-4)^2+16y^2+200(-4)-160y+400=0$

$\cancel{400}+16y^2-\cancel{800}-160y+\cancel{400}=0$

$16y(y-10)=0$

$y=0 \quad y=10$ $\qquad (-4,0)\ (-4,10)$

Vert: $32y-160=0$ $\qquad$ (Denom $=0$)

$32y=160$

$y=5$

$25x^2+16(5)^2+200x-160(5)+400=0$

$25x^2+\cancel{400}+200x-\cancel{800}+\cancel{400}=0$

$25x(x+8)=0$

$x=0 \quad x=-8$ $\qquad (0,5)\ (-8,5)$

p144 Section 2.6: Related Rates

- Find a related rate
- Use related rates to solve real-life problems

Finding Related Rates
We have used the chain rule to find dy/dx implicitly, but you can also use the chain rule to find the rates of change of two or more related variables that are changing with respect to *time*.

Example 1: Two Rates that are Related
Suppose x & y are both differentiable functions of t and are related by the equation $y = x^2 + 3$. Find dy/dt when x = 1, given that dx/dt = 2 when x = 1.

$$y = x^2 + 3$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

Subst

$$\frac{dy}{dt} = 2(1)(2) = 4$$

(2) $y = 2(x^2 - 3x)$

$$y = 2x^2 - 6x$$

$$\frac{dy}{dt} = (4x - 6)\frac{dx}{dt}$$

(a) Find $\frac{dy}{dt}$ $x = 3$ $\frac{dx}{dt} = 2$

$$\frac{dy}{dt} = (4(3) - 6)(2) = 12$$

(b) Find $\frac{dx}{dt}$ $x = 1$ $\frac{dy}{dt} = 5$

$$5 = (4(1) - 6)\frac{dx}{dt}$$

$$5 = -2\frac{dx}{dt}$$

$$-\frac{5}{2} = \frac{dx}{dt}$$

(6) $y = \dfrac{1}{1+x^2}$

$y = (1+x^2)^{-1}$

$\dfrac{dy}{dt} = -1(1+x^2)^{-2}(2x)\dfrac{dx}{dt}$

$\dfrac{dy}{dt} = \dfrac{-2x}{(1+x^2)^2}\dfrac{dx}{dt}$ $\qquad \dfrac{dx}{dt} = 2$ cm/sec

Find $\dfrac{dy}{dt}$

(a) $x=-2$: $\dfrac{dy}{dt} = \dfrac{-2(-2)}{(1+(-2))^2}(2) = \dfrac{8}{25}$ cm/sec

(b) $x=0$: $\dfrac{dy}{dt} = \dfrac{-2(0)}{(1+(0))^2}(2) = 0$ cm/sec

(8) $y = \sin x$

$\dfrac{dy}{dt} = \cos x \dfrac{dx}{dt}$ $\qquad \dfrac{dx}{dt} = 2$ cm/sec

Find $\dfrac{dy}{dt}$

(a) $x=\frac{\pi}{6}$: $\dfrac{dy}{dt} = \cos\frac{\pi}{6}(2) = \frac{\sqrt{3}}{2}(2) = \sqrt{3}$ cm/s

(b) $x=\frac{\pi}{4}$: $\dfrac{dy}{dt} = \cos\frac{\pi}{4}(2) = \frac{\sqrt{2}}{2}(2) = \sqrt{2}$ cm/se

Example 2: Ripples in a Pond p145

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(4)(1) = 8\pi$ ft²/sec

Rate of Change of radius:

$\frac{dr}{dt} = 1$ ft/sec

$r = 4$ Find $\frac{dA}{dt}$

Guidelines For Solving Related-Rate Problems

1. Identify all given quantities and quantities to be determine. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation with respect to time t.
4. After completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Here are some examples of mathematical models involving rates of change.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour	x = distance traveled dx/dt = 50 when t = 1
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour	V = volume of water in pool dV/dt = 10 m3/hr
A gear is revolving at a rate of 25 revolutions per minute (1 revolution = 2πrad)	θ = angle of revolution dθ/dt = 25(2π) rad/min

Example 3: An Inflating Balloon p146

Air is being pumped into a spherical balloon at a rate of 4.5 cubic feet per minute. Find the rate of change of the radius when the radius is 2 feet.

V = volume   r = radius

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad r=2 \qquad \frac{dV}{dt} = \frac{9}{2}$$

$$\frac{9}{2} = 4\pi(2)^2 \frac{dr}{dt}$$

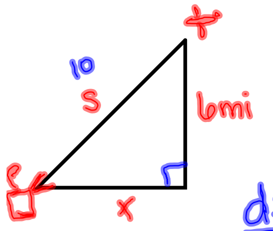
$$\frac{9}{2} = 16\pi \frac{dr}{dt}$$

$$\frac{9}{32\pi} \approx 0.0895 \approx \frac{dr}{dt}$$

9/(32π)
.0895246555

HW p149 # 3, 5, 15, 19

Example 4: The Speed of an Airplane Tracked by Radar
An airplane is flying on a flight path that will take it directly over a radar tracking station. If ~~y~~ s is decreasing at a rate of 400 miles per hour when s = 10 miles, what is the speed of the plane?



When $\underline{s=10}$: $x^2+(6)^2=(10)^2$

$x^2=100-36$

$\sqrt{x^2}=\sqrt{64}$

$\underline{x=8}$

$\frac{ds}{dt}=-400\text{mi/hr}$

$$x^2+6^2=s^2$$

$$2x\frac{dx}{dt}=2s\frac{ds}{dt}$$

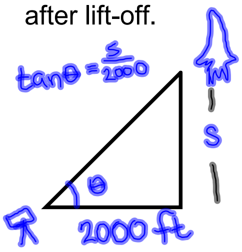
$s=10,\ x=8,\ \frac{ds}{dt}=-400$

$$2(8)\frac{dx}{dt}=2(10)(-400)$$

$$\frac{dx}{dt}=-500\text{ mi/hr}$$

speed = 500 mi/hr

Example 5: A Changing Angle of Elevation
Find the rate of change in the angle of elevation of the camera at 10 seconds after lift-off.



$s=50t^2$ $t=10$

$s=50(10)^2=5000$

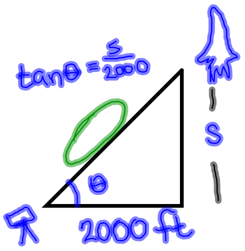
$\frac{ds}{dt}=100t \rightarrow$ velocity of rocket

Find $\frac{d\theta}{dt}$

$$\tan\theta=\frac{s}{2000}=\frac{1}{2000}s$$

$$\frac{\sec^2\theta\frac{d\theta}{dt}}{\sec^2\theta}=\frac{\frac{1}{2000}\boxed{\frac{ds}{dt}}}{\sec^2\theta}$$

$$\frac{d\theta}{dt}=\frac{\cos^2\theta}{2000}\cdot 100t$$



$a^2+b^2=c^2$

$2000^2+s^2=c^2$

$\sqrt{2000^2+s^2}=c$

$\cos\theta=\frac{a}{h}=\frac{2000}{\sqrt{2000^2+s^2}}$

$$\frac{d\theta}{dt}=\cos^2\theta\ \frac{t}{20}$$

$$\frac{d\theta}{dt}=\left(\frac{2000}{\sqrt{2000^2+s^2}}\right)^2\left(\frac{t}{20}\right)$$

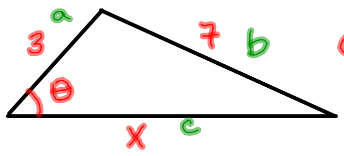
$t=10$
$s=5000$

$$\frac{d\theta}{dt}=\left(\frac{2000}{\sqrt{2000^2+5000^2}}\right)^2\frac{10}{20}=\frac{2}{29}$$

```
(2000/√(2000²+50
00²))²*(1/2)▸Fra
c
                2/29
```

Example 6: The Velocity of a Piston

In an engine a 7-inch connecting rod is fastened to a crank of radius 3 inches. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. Find the velocity of the piston when θ = π/3.



$\frac{d\theta}{dt} = 200(2\pi)$ $\quad$ 1 Rev = 2π

$\frac{d\theta}{dt} = 400\pi$ $\quad$ Find $\frac{dx}{dt}$ $\quad \theta = \frac{\pi}{3}$

Law of cosines: $b^2 = a^2 + c^2 - 2ac\cos\theta$

$$(7)^2 = (3)^2 + x^2 - 2(3)x\cos\theta$$

$$49 = 9 + x^2 - 6x\cos\theta$$

Derivative

$$0 = 2x\frac{dx}{dt} - 6\left[x(-\sin\theta)\frac{d\theta}{dt} + \cos\theta\,(1)\frac{dx}{dt}\right]$$

$$0 = 2x\frac{dx}{dt} + 6x\sin\theta\frac{d\theta}{dt} - 6\cos\theta\frac{dx}{dt}$$

$$(-2x + 6\cos\theta)\frac{dx}{dt} = 6x\sin\theta\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{6x\sin\theta}{-2x+6\cos\theta}\frac{d\theta}{dt}$$

$\theta = \frac{\pi}{3}$ $\quad \frac{d\theta}{dt} = 400\pi$

Law of cosines: $b^2 = a^2 + c^2 - 2ac\cos\theta$

$$7^2 = 3^2 + x^2 - 2(3)(x)\cos\frac{\pi}{3}$$

$$49 = 9 + x^2 - 6x\left(\tfrac{1}{2}\right)$$

$$40 = x^2 - 3x$$

$$0 = x^2 - 3x - 40$$

$$0 = (x-8)(x+5)$$

$$x = 8$$

$\theta = \frac{\pi}{3}$ $\quad \frac{d\theta}{dt} = 400\pi$

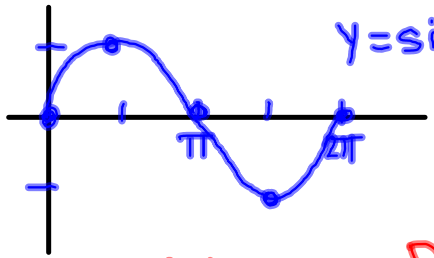
$$\frac{dx}{dt} = \frac{6x\sin\theta}{-2x+6\cos\theta}\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{6(8)\sin\frac{\pi}{3}}{-2(8)+6\cos\frac{\pi}{3}}(400\pi) = \frac{48\left(\frac{\sqrt{3}}{2}\right)}{-16+6\left(\frac{1}{2}\right)}(400\pi)$$

$$= \frac{9600\pi\sqrt{3}}{-13} = -4018 \text{ in/min}$$

HW p149 #31, 35

14. Find the rate of change of the distance between the origin and a moving point on the graph of y = sin x if dx/dt = 2 centimeters per second.



$y = \sin x$ $\quad \frac{dx}{dt} = 2 \text{cm/sec}$

Distance: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Dist b/n origin (0,0) & moving pt (x,y)

$D = \sqrt{x^2 + y^2}$ subst $y = \sin x$

$D = \sqrt{x^2 + \sin^2 x} = (x^2 + \sin^2 x)^{1/2}$ Deriv.

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2\sin x \cos x)\frac{dx}{dt}$$

$$\frac{dD}{dt} = \frac{\cancel{2}(x + \sin x \cos x)}{\cancel{2}(x^2 + \sin^2 x)^{1/2}} \frac{dx}{dt}$$ Let $\frac{dx}{dt} = 2$

$$\frac{dD}{dt} = \left(\frac{x + \sin x \cos x}{\sqrt{x^2 + \sin^2 x}}\right)(2) = \frac{2x + 2\sin x \cos x}{\sqrt{x^2 + \sin^2 x}}$$

HW p150 #27

ⓐ $a^2 + b^2 = c^2$

ⓑ $A = \frac{1}{2}bh$

ⓒ Angle - trig sohcahtoa

18. Volume: The radius r of a sphere is increasing at a rate of 2 inches per minute.

(a) Find the rate of change of the volume when r = 6 inches and r = 24 inches

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dr}{dt} = 2 \text{ in/min}$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2)\frac{dr}{dt}$$

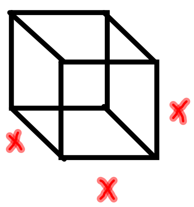
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

ⓐ $r=6: \frac{dV}{dt} = 4\pi(6)^2(2) = 288\pi$ in³/min

$r=24: \frac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi$ in³/min

ⓑ If $\frac{dr}{dt}$ is constant, $\frac{dV}{dt}$ is proportional to r^2

20. Volume: All edges of a cube are expanding at a rate of 3 centimeters per second. How fast is the volume changing when each edge is
(a) 1 centimeter and (b) 10 centimeters?



$$V = x^3 \qquad \frac{dx}{dt} = 3 \text{ cm/sec}$$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

ⓐ $x=1: \frac{dV}{dt} = 3(1)^2(3) = 9$ cm³/sec

ⓑ $x=10: \frac{dV}{dt} = 3(10)^2(3) = 900$ cm³/sec

22. Volume: The formulas for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$

Find the rate of change of the volume if dr/dt is 2 inches per minute and h = 3r when (a) r = 6 inches and (b) r = 24 inches.

$V = \frac{1}{3}\pi r^2 h$ Subst $h = 3r$

$V = \frac{1}{\cancel{3}}\pi r^2(\cancel{3}r)$

$V = \pi r^3$

$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

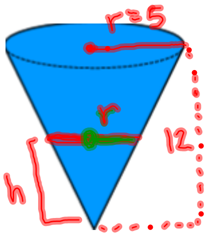
(a) $r = 6$: $\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi$ in^3/min

(b) $r = 24$: $\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi$ in^3/min

24. Depth: A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

$r = 5$

h



$\frac{dV}{dt} = 10$ ft^3/min

Similar Δs: $\frac{r}{5} = \frac{h}{12} \rightarrow r = \frac{5}{12}h$

$V = \frac{1}{3}\pi r^2 h$ Subst: $r = \frac{5}{12}h$

$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h$

$V = \frac{25\pi}{3(144)} h^3$

$\frac{dV}{dt} = \frac{25\pi}{\cancel{3}(144)}(\cancel{3}h^2)\frac{dh}{dt}$

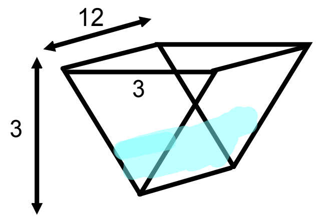
$\frac{dV}{dt} = \frac{25\pi h^2}{144} \cdot \boxed{\frac{dh}{dt}}$

$144\left(10 = \frac{25\pi(8)^2}{\cancel{144}} \frac{dh}{dt}\right)\cancel{144}$

$1440 = 1600\pi \frac{dh}{dt}$

$\frac{9}{10\pi}$ ft/min $= \frac{dh}{dt}$

26. Depth: A trough is 12 feet long and 3 feet across the top. It ends are isosceles triangles with altitudes of 3 feet.
(a) If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when it is 1 foot deep?
(b) If the water is rising at a rate of 3/8 inch per minute when h = 2, determine the rate at which water is being pumped into the trough.



$\frac{dV}{dt} = 2 ft^3/min$

$V = \frac{1}{2}bh(12)$

$V = 6bh$ $\quad b = h$

$V = 6h^2$

$\frac{dV}{dt} = 12h\boxed{\frac{dh}{dt}}$

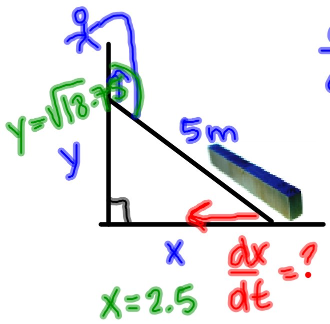
ⓐ $2 = 12(1)\frac{dh}{dt}$

$\frac{1}{6} ft/min = \frac{dh}{dt}$

ⓑ $\frac{dh}{dt} = \frac{3}{8} in/min \quad h = 2$

$\frac{dV}{dt} = 12(2)\left(\frac{3}{8}\right) = 9 ft^3/min$

28. Construction: A construction worker pulls a 5-meter plank up the side of a building under construction by means of a rope tied to one end of the plank. Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope at a rate of 0.15 meters per second. How fast is the end of the plank sliding along the ground when it is 2.5 meters from the wall of the building?



$\frac{dy}{dt} = 0.15 m/s$

$x^2 + y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

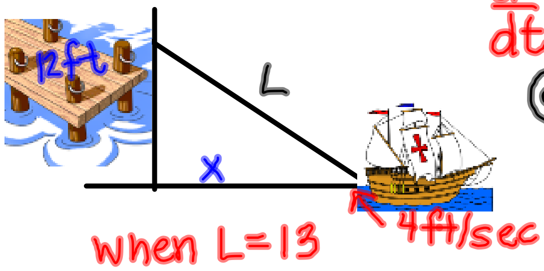
$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$

$\frac{dx}{dt} = -\frac{y}{x}(0.15)$

$\frac{dx}{dt} = \frac{-\sqrt{18.75}}{2.5}(0.15) = -0.269$ m/sec

30. Boating: A boat s pulled into a dock by means of a winch 12 feet above the deck of the boat. (let L = length of the rope)
(a) The winch pulls in rope at a rate of 4 feet per second. Determine the speed of the boat when there is 13 feet of rope out. What happens to the speed of the boat as it gets closer to the dock?
(b) Suppose the boat is moving at a constant rate of 4 feet per second. Determine the speed at which the winch pulls rope when there is a total of 13 feet of rope out. What happens to the speed at which the winch pulls in rope as the boat gets closer to the dock?

$\frac{dL}{dt} = -4$ ft/sec (L gets smaller)



when $L=13$

$13^2 = 12^2 + x^2$

$5 = x$

ⓐ $L^2 = (12)^2 + x^2$

$L^2 = 144 + x^2$

$2L\frac{dL}{dt} = 2x\frac{dx}{dt}$

$\frac{L}{x}\frac{dL}{dt} = \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{L}{x}(-4) = -\frac{4L}{x}$

$\frac{dx}{dt} = \frac{-4(13)}{5} = -10.4$ ft/sec

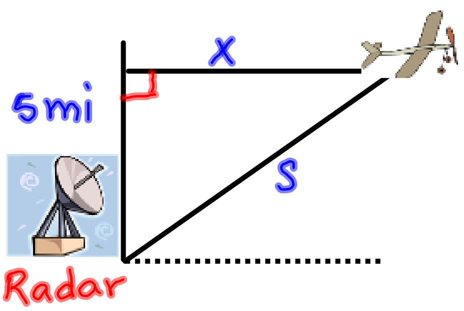
Therefore the speed of the boat increases as it approaches the dock.

ⓑ $\frac{dx}{dt} = -4$ ft/sec $\quad L = 13$ ft

$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt} = \frac{5}{13}(-4) = -1.538$ ft/sec

As L approaches zero, dL/dt increases

32. Air Traffic Control: An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna. When the plane is 10 miles away (s = 10), the radar detects that the distance s is changing at a rate of 240 mile per hour. What is the speed of the plane?



When $s = 10$

$5^2 + x^2 = 10^2$

$x = \sqrt{100-25}$

$= 5\sqrt{3}$

$\frac{ds}{dt} = 240 \text{ mi/hr}$

$x^2 + y^2 = s^2$ $\qquad y = 5 \text{ mi}$

$x^2 + 25 = s^2$

$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}$

$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(240) = 277.128 \text{ mi/hr}$

Speed = 277.128 mph