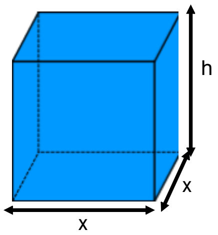
p211 Section 3.7: Optimization Problems
In other words: Applied Max & Min Problems

WARM-UP: p212 - find the volume of each of the 5 figures

Example 1: Finding Maximum Volume
A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



Surface Area = 108 in²

$S = \text{Area Base} + 4\ \text{Area Sides} = x^2 + 4xh$

Max Volume: $V = x^2 h$

Primary Equation

Solve S equat for h (to get rid of a variable)

$108 = x^2 + 4xh$

$h = \frac{108 - x^2}{4x}$ Subst h into V

$$V = x^2\left(\frac{108 - x^2}{4x}\right) = \frac{108x^2}{4x} - \frac{x^4}{4x} = 27x - \frac{x^3}{4}$$

Domain: $x > 0,\ V > 0$

Area of Base $(x^2) \le 108$

$0 < x^2 \le 108$

$0 < x \le \sqrt{108}\ (10.39)$

To maximize V, find crit #

$\frac{dV}{dx} = 27 - \frac{3x^2}{4} = 0$

$27 = \frac{3x^2}{4}$

$36 = x^2$

$x = \pm 6 \rightarrow x = 6$ (domain)

$V(0) = 0$ $\qquad V(6) = 108$ $\qquad V(\sqrt{108}) = 0$

↳ max

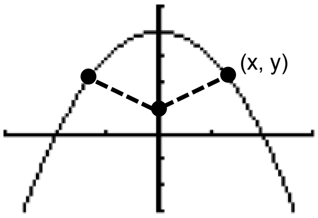
$h = \frac{108 - x^2}{4x} = 3$ $\qquad$ Dimensions: $6 \times 6 \times 3$ in

Guidelines for Solving Applied Minimum and Maximum Problems

1. Identify *all given quantities* and *quantities to be determined*. When feasible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized (or minimized).
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.4.

Example 2: Finding Minimum Distance

Which points on the graph of the indicated function are closest to the point (0, 2)?



$f(x) = 4 - x^2$

Distance b/n $(\overset{x_1}{0}, \overset{y_1}{2})$ & (x, y)

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(x - 0)^2 + (y - 2)^2}$ $\quad y = 4 - x^2$

$d = \sqrt{x^2 + [(4 - x^2) - 2]^2} = \sqrt{x^2 + (2 - x^2)^2}$

$= \sqrt{x^2 + 4 - 4x^2 + x^4} = \sqrt{x^4 - 3x^2 + 4}$ $\quad$ D = All Real #s

* you only need to find the critical #s of inside the radical

$f(x) = x^4 - 3x^2 + 4$

$f'(x) = 4x^3 - 6x = 0$

$2x(2x^2 - 3) = 0$

$x = 0 \quad x = \pm\sqrt{\frac{3}{2}}$

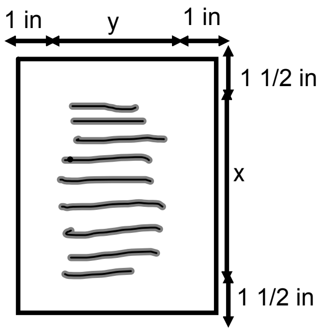
$-\infty < x < -\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}} < x < 0$	$0 < x < \sqrt{\frac{3}{2}}$	$\sqrt{\frac{3}{2}} < x < \infty$
$x = -2$	$x = -1$	$x = 1$	$x = 2$
$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$

$\cup$ $x = -\sqrt{\frac{3}{2}}$ $\quad \cap \quad$ $\cup$ $x = \sqrt{\frac{3}{2}}$

Minimum: $\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ $\left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$

Example 3: Finding Minimum Area

A rectangular page is to contain 24 square inches of print. The margins at the tops and bottom of the page are to be a 1 1/2 inches, and the margins on the left and right are to be 1 inch. What should the dimensions ~~of~~ should the page be so that the least amount of paper is used?



Area Minimized: $A = lw$

$A = (x+3)(y+2)$

Printed Area $A = 24 = xy$

Solve for y: $y = \frac{24}{x}$

Domain: $x > 0$

$A = (x+3)\left(\frac{24}{x}+2\right)$ FOIL

$A = 24 + 2x + \frac{72}{x} + 6 = 30 + 2x + \frac{72}{x}$

$\frac{72}{x} \to 72x^{-1}$

$\frac{dA}{dx} = 2 - 72x^{-2} = 2 - \frac{72}{x^2} = 0$

$2 = \frac{72}{x^2}$

$2x^2 = 72$

$x^2 = 36$

$x = \pm 6 \to x = 6$

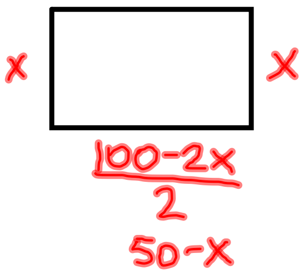
$0 < x < 6$	$6 < x < \infty$
$x = 1$	$x = 7$
$f' < 0$	$f' > 0$

min

$\therefore x = 6 \quad y = \frac{24}{6} = 4$

$x + 3 = 9 \quad y + 2 = 6$

100 feet of wire will be used to enclose a rectangular area. What are the dimensions of the rectangle of the largest area?

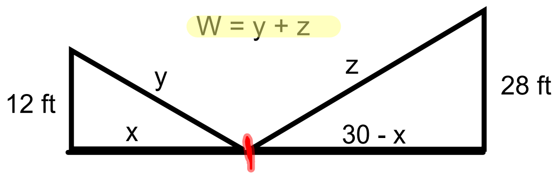


$P = 100$

25×25

Example 4: Finding Minimum Length

Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least wire?



Pyth Thrm:

$$x^2 + 12^2 = y^2$$

$$\sqrt{x^2 + 144} = y$$

$$(30-x)^2 + 28^2 = z^2$$

$$900 - 60x + x^2 + 784 = z^2$$

$$x^2 - 60x + 1684 = z^2$$

$$\sqrt{x^2 - 60x + 1684} = z$$

$$W = y + z$$

$$W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

$$W = (x^2 + 144)^{1/2} + (x^2 - 60x + 1684)^{1/2}$$

$$\frac{dW}{dx} = \frac{1}{\cancel{2}}(x^2+144)^{-1/2}(\cancel{2}x) + \frac{1}{\cancel{2}}(x^2-60x+1684)^{-1/2}\overset{\cancel{2}(x-30)}{(2x-60)}$$

$$0 = \frac{x}{\sqrt{x^2+144}} + \frac{x-30}{\sqrt{x^2-60x+1684}}$$

$$\frac{-x}{\sqrt{x^2+144}} = \frac{x-30}{\sqrt{x^2-60x+1684}}$$

cross - multiply

$$\left[-x\sqrt{x^2-60x+1684} = (x-30)\sqrt{x^2+144}\right]^2$$

$$x^2(x^2-60x+1684) = (x-30)^2(x^2+144)$$

$$\cancel{x^4}-\cancel{60x^3}+1684x^2 = \cancel{x^4}-\cancel{60x^3}+900x^2+144x^2-8640x+129{,}600$$

$$0 = -640x^2-8640x+129{,}600$$

Divide by -320

$$0 = 2x^2+27x-405$$

$$\begin{array}{c|c|c|} & 2x & 45 \\ \hline x & 2x^2 & 45x \\ \hline -9 & -18x & -405 \\ \hline \end{array}$$

$$2(-405) = -810$$

$$\underline{45x} - \underline{18x} = 27x$$

$$0 = (2x+45)(x-9)$$

$0 < x < 30$

$$\cancel{x = \frac{-45}{2}} \qquad x = 9$$

$$0 < x < 9 \qquad 9 < x < 30$$

$$f' < 0 \qquad f' > 0$$

Min $x = 9$

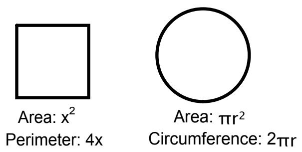
Stake should be 9 ft from 12 ft pole

In each of the first four examples, the extreme value occurs at a critical number. Although this happens often, remember that an extreme value can also occur at an endpoint of an interval

Example 5: An Endpoint Maximum

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Total Area = Area Square + Area Circle



$$A = x^2 + \pi r^2$$

4 Ft wire

$4 =$ Perim Sqr $+$ Circ Circle

D: $0 < x < 1$

$$4 = 4x + 2\pi r$$

Solve for r

$$\frac{4-4x}{2\pi} = r = \frac{2-2x}{\pi}$$

$$A = x^2 + \pi r^2$$

$$A = x^2 + \pi\left(\frac{2-2x}{\pi}\right)^2$$

$$A = x^2 + \cancel{\pi}\left(\frac{4-8x+4x^2}{\pi^{\cancel{2}}}\right) = \frac{\pi}{\pi}x^2 + \frac{4}{\pi} - \frac{8x}{\pi} + \frac{4x^2}{\pi}$$

$$A = \frac{1}{\pi}\left[(\pi+4)x^2 - 8x + 4\right]$$

$$\frac{dA}{dx} = \frac{1}{\pi}\left[2(\pi+4)x - 8\right] = 0$$

$$2(\pi+4)x = 8$$

$$(\pi+4)x = 4$$

$$x = \frac{4}{\pi+4} \approx 0.560$$

$0 < x < 0.560$ | $f' < 0$

$0.560 < x < 1$ | $f' > 0$

Min

$A(0) = 1.273$ Max

$A(0.560) = 0.56$

$A(1) = 1$

Max Area length of square $= 0$
All wire used for circle

HW p216 # 18, 20, 23, 30