p242 Section 4.1: Antiderivatives and Indefinite Integration

Warm-Up: Exploration on page 242

Definition of an Antiderivative

A function F is an antiderivative of f on an interval I if F '(x) = f(x) for all x in I

Theorem 4.1: Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form (for all x in I & C is a constant)

$$G(x) = F(x) + C$$

** An family of antiderivatives of a function can be represented by adding a constant to a known antiderivative.
** C is called the constant of integration
** A family of functions can be represented by the general antiderivative of f, G.
**A differential equation in x & y is an equation that involves x, y, and derivatives of y (the equation is the derivative - the answer is the original function)

Example 1: Solving a Differential Equation

Find the general solution of the differential equation y' = 2

** Find a function whose derivative is 2 then write the general solution (add C)

$y' = 2 \rightarrow y = 2x + C$

Notation for Antiderivatives

When solving a differential equation of the form, $\frac{dy}{dx} = f(x)$ it is convenient to write it in the equivalent differential form $dy = f(x)dx$

The operation of finding all solutions of this equation is called antidifferentiation (or indefinite integration) and is denoted by an integral sign $\int$. The general solution is denoted by:

Variable of integration

Constant of integration

$$y = \int f(x)dx = F(x) + C$$

Integrand

The expression $\int f(x)dx$ is read as the antiderivative of f with respect to x. The differential dx serves to identify x as the variable of integration. The term indefinite integral is a synonym for antiderivative

Basic Integration Rules

Integration is the "inverse" of differentiation - Differentiation is the "inverse" of integration

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C] = 0$	$\int 0dx = C$
$\frac{d}{dx}[kx] = k$	$\int kdx = kx + C$ $\int 2\,dx = 2x + c$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)dx = k\int f(x)dx$ $\int 3f(x)\,dx = 3\int f(x)\,dx$

Differentiation Formula | Integration Formula

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \qquad \int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\frac{d}{dx}[\sin x] = \cos x \qquad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \int \sec^2 x dx = \tan x + C$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \int \sec x \tan x dx = \sec x + C$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \int \csc^2 x dx = -\cot x + C$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x \qquad \int \csc x \cot x dx = -\csc x + C$$

Example 2: Applying the Basic Integration Rules

Describe the antiderivatives of 3x

$$\int 3x\,dx = 3\int x\,dx = 3\int x^1\,dx$$

$$= 3\left(\frac{x^2}{2}\right) + C = \frac{3}{2}x^2 + C$$

Example 3: Rewriting Before Integrating

Original Integral	Rewrite	Integrate	Simplify
$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

** You can check your answer by finding the derivative!!

Example 4: Integrating Polynomial Functions

a $\int dx = \int 1 dx$

$= x + C$

b $\int (x+2)dx = \int x dx + \int 2 dx$

$= \frac{x^2}{2} + C_1 + 2x + C_2 = \frac{x^2}{2} + 2x + C$

c $\int (3x^4 - 5x^2 + x)dx = 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C = \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$

Example 5: Rewriting Before Integrating

$$\int \frac{x+1}{\sqrt{x}}dx = \int\left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)dx = \int\left(\frac{x}{x^{1/2}} + \frac{1}{x^{1/2}}\right)dx = \int (x^{1/2} + x^{-1/2})dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

** Do not integrate the numerator and denominator separately!!

$$\int \frac{x+1}{\sqrt{x}}dx \neq \frac{\int (x+1)dx}{\int \sqrt{x}dx}$$

Example 6: Rewriting Before Integrating

$$\int \frac{\sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx = \int \sec x \tan x dx = \sec x + C$$

Find the indefinite integral and check the result with differentiation

#23. $\int \sqrt[3]{x^2} dx = \int x^{2/3} dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5} x^{5/3} + C$

Check: $y' = \frac{3}{5}\left(\frac{5}{3}\right) x^{2/3} = x^{2/3}$

#25. $\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$

Check: $y' = -\frac{1}{2}(-2x^{-3}) = \frac{1}{x^3}$

#29. $\int (x+1)(3x-2) dx = \int (3x^2 + x - 2) dx$

$$= 3\frac{x^3}{3} + \frac{x^2}{2} - 2x + C = x^3 + \frac{1}{2}x^2 - 2x + C$$

#31. $\int y^2 \sqrt{y} dy = \int y^2 y^{1/2} dy = \int y^{5/2} dy = \frac{y^{7/2}}{7/2} + C = \frac{2}{7} y^{7/2} + C$

#37. $\int (1-\csc t \cot t)dt = t + \csc t + C$

#39. $\int (\sec^2 \theta - \sin \theta)d\theta = \tan \theta + \cos \theta + C$

#41. $\int (\tan^2 y + 1)dy = \int \sec^2 y dy = \tan y + C$

Initial Conditions and Particular Solutions - Differential Equations

Graphs of any two antiderivatives of f are vertical translations of each other.

In many applications of integration, enough information is given to find a **particular solution** - but you need the vale of y = F(x) for one value of x (an **initial condition**)

Example 7: Finding a Particular Solution

Find the general solution and use it to find the particular solution that satisfies the initial condition F(1)=0

$$F'(x) = \frac{1}{x^2}, x > 0$$

$$F(x) = \int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

Now substitute for x = 1 and y = 0 (the initial condition) and solve for C

$$-\frac{1}{1} + C = 0$$

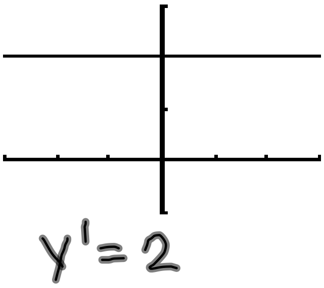
$$C = 1$$

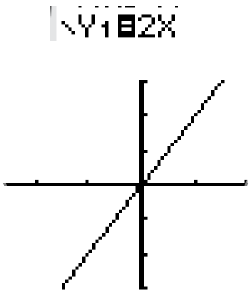
Now substitute the value for C into the general solution to write the particular solution

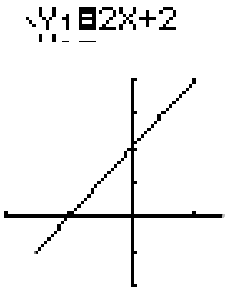
$$F(x) = -\frac{1}{x} + 1, x > 0$$

The graph of the derivative of a function is given. Sketch the graphs of two functions that have the given derivative.

#45.







Find the equation for y, given the derivative and the indicated point on the curve.

#49.

$$\frac{dy}{dx} = 2x - 1$$

Point: (1, 1)

$$y = \int (2x-1)dx = 2\frac{x^2}{2} - x + C$$

$$y = x^2 - x + C$$

Now substitute x = 1 and y = 1 and solve for C

$$1 = (1)^2 - (1) + C$$
$$1 = C$$
$$y = x^2 - x + 1$$

Solve the differential equation

#57. $h'(t) = 8t^3 + 5$ h(1) = -4

$$h(t) = \int (8t^3 + 5)dt = 8\frac{t^4}{4} + 5t + C$$

$$h(t) = 2t^4 + 5t + C$$

$$-4 = 2(1)^4 + 5(1) + C$$

$$-4 = 7 + C$$

$$-11 = C$$

$$h(t) = 2t^4 + 5t - 11$$

#59. $f''(x) = 2$ f '(2) = 5 f(2) = 10

$$f'(x) = \int 2dx$$

$$f'(x) = 2x + C$$

$$5 = 2(2) + C$$

$$1 = C$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1)dx = 2\frac{x^2}{2} + x + c$$

$$f(x) = x^2 + x + C$$

$$10 = (2)^2 + (2) + C$$

$$4 = C$$

$$f(x) = x^2 + x + 4$$

Example 8: Solving a Vertical Motion Problem

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet.

a. Find the position function giving the height s as a function of the time t.

b. When does the ball hit the ground?

** Original Function = position function
First Derivative = Velocity Function
Second Derivative = Acceleration Function

a. Let t = 0 represent the initial time. The two given initial conditions can be written as follows:

s(0) = 80 (initial height is 80 feet)
s '(0) = 64 (initial velocity is 64 feet per second)
** Using -32 feet per second squared as the acceleration due to gravity, write a differential equation

$$s''(t) = -32$$

$$s'(t) = \int -32\,dt$$

$$s'(t) = -32t + C_1$$

$$64 = -32(0) + C_1$$

$$64 = C_1$$

$$s'(t) = -32t + 64$$ Velocity Function

Now integrate the velocity function to find the position function

$$s(t) = \int (-32t + 64)dt = -32\left(\frac{t^2}{2}\right) + 64t + C_2$$

$$s(t) = -16t^2 + 64t + C_2$$

$$80 = -16(0)^2 + 64(0) + C_2$$

$$80 = C_2$$

$$s(t) = -16t^2 + 64t + 80$$

b. When does the ball hit the ground? In other words, when does s(t) = 0?

$$0 = -16t^2 + 64t + 80$$

$$0 = -16(t^2 - 4t - 5)$$

$$0 = -16(t - 5)(t + 1)$$

t = 5 and t = -1 **but t ≠ -1 because time cannot be negative

$\therefore$ It took 5 seconds for the ball to hit the ground

HW: p249 #16 - 40 e/o/e (every other even); 52, 56, 60, 70, 72