p324 Section 5.2: The Natural Logarithmic Function: Integration

Theorem 5.5: Log Rule for Integration

Let u be a differentiable function of x

1. $\int \frac{1}{x}\,dx = \ln|u| + C$

2. $\int \frac{1}{u} = \ln|u| + C$

$$\int \frac{u'}{u} = \ln|u| + C$$

Example 1: Using the Log Rule for Integration

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx = 2\ln|x| + C = \ln(x^2) + C$$

** Note: Since x^2 cannot be negative the absolute value symbol is not needed

Example 2: Using the Log Rule with a Change of Variables

$$\int \frac{1}{4x-1}\,\underline{dx}$$

$$= \int \frac{1}{u}\left(\frac{du}{4}\right) = \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|4x-1| + C$$

$u = 4x - 1$

$du = \boxed{4}\,\underline{dx}$

$\frac{du}{4} = dx$

(5).

$$\int \frac{1}{3-2x}\underline{dx} = \int \frac{1}{u}\left(\frac{du}{-2}\right) = -\frac{1}{2}\int \frac{1}{u}du$$

$$= -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|3-2x| + C$$

$u = 3-2x$

$du = \boxed{-2}\underline{dx}$

$\frac{du}{-2} = dx$

(9).

$$\int \frac{x^2-4}{x}dx = \int \frac{x^2}{x} - \frac{4}{x}dx = \int x - \frac{4}{x}dx$$

$$= \frac{x^2}{2} - 4\ln|x| + C$$

(11).

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}dx = \int \frac{1}{u}\left(\frac{du}{3}\right) = \frac{1}{3}\int \frac{1}{u}du$$

$$= \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|x^3+3x^2+9x\right| + C$$

$u = x^3+3x^2+9x$

$du = 3x^2+6x+9dx$

$du = 3(x^2+2x+3)dx$

$\frac{du}{3} = x^2+2x+3dx$

(37). Solve the differential equation through the point, (1, 0)

$$\frac{dy}{dx}=\frac{3}{2-x}$$

$$y=\int\frac{3}{2-x}dx=-3\int\frac{1}{x-2}dx$$

$$y=-3\ln|x-2|+C$$

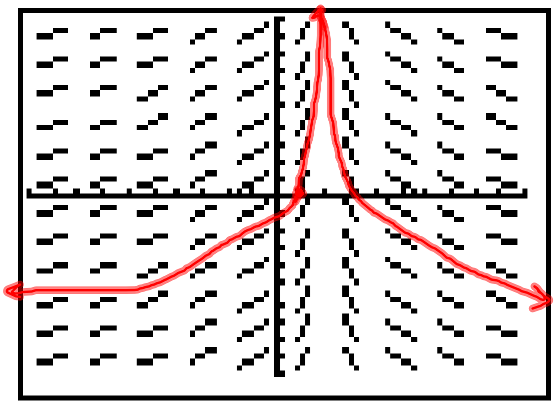
$$(1,0):0=-3\ln|1-2|+C$$

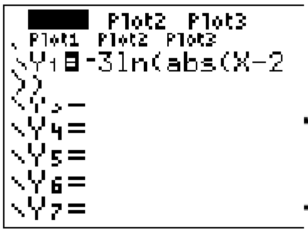
$$0=-3\ln|-1|+C$$

$$0=-3\ln 1+C$$

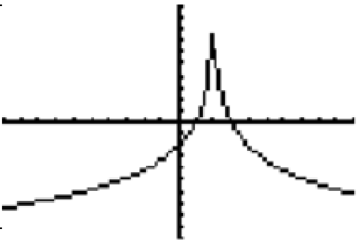
$$0=C$$

$$y=-3\ln|x-2|$$





```
Plot1 Plot2 Plot3
\Y1=-3ln(abs(X-2
))
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



(41). A differential equation, a point **(0, 1)** and a slope field are given.

(a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point.

(b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).

$$\frac{dy}{dx} = \frac{1}{x+2}$$

$$y = \int \frac{1}{x+2}\,dx = \ln|x+2| + C$$

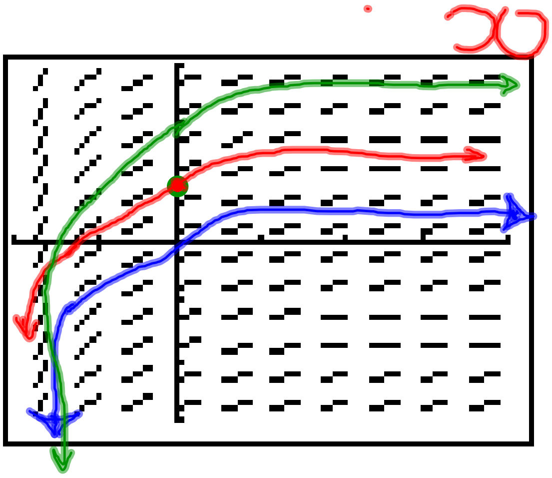
$$1 = \ln|0+2| + C$$

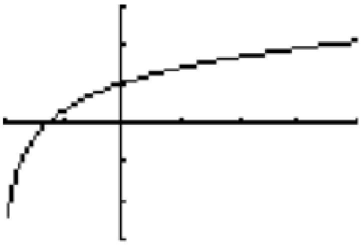
$$1 = \ln 2 + C$$

$$1 - \ln 2 = C$$

$$y = \ln|x+2| + 1 - \ln 2$$

$$y = \ln\left|\frac{x+2}{2}\right| + 1$$





(43). Evaluate the definite integral. Use a graphing utility to verify your result

$$\int_0^4 \frac{5}{3x+1}dx = 5\int_0^4 \frac{1}{u}\left(\frac{du}{3}\right) = \frac{5}{3}\int_0^4 \frac{1}{u}du$$

$$= \frac{5}{3}\ln|u|\Big]_0^4 = \frac{5}{3}\ln|3x+1|\Big]_0^4$$

$$= \frac{5}{3}\ln|3(4)+1| - \frac{5}{3}\ln|3(0)+1|$$

$$= \frac{5}{3}\ln|13| - \frac{5}{3}\ln|1|$$

$$= \frac{5}{3}\ln\frac{13}{1} = \frac{5}{3}\ln 13 \approx 4.275$$

$u = 3x+1$

$du = 3dx$

$\frac{du}{3} = dx$

**Instead of finding new upper and lower limits, I re-substituted for u after I integrated the problem

Verify in the calculator:

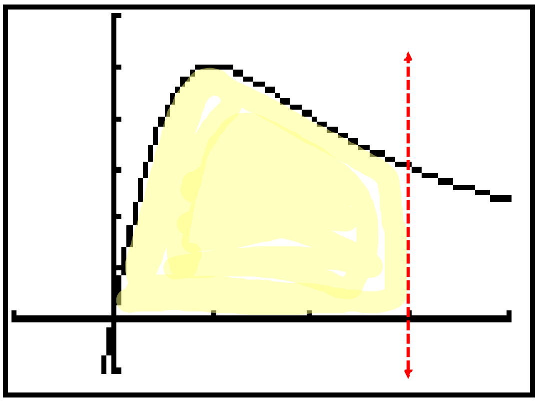
```
fnInt(5/(3X+1),X
,0,4)
          4.274915596
```

Example 3: Finding Area with the Log Rule

Find the area of the region bounded by the x-axis, the line x = 3 and the graph of

$$y = \frac{x}{x^2+1}$$

```
WINDOW
 Xmin=-1
 Xmax=4
 Xscl=1
 Ymin=-.1
 Ymax=.6
 Yscl=.1
 Xres=1
```



Using the graph, what values should we use for the lower and upper bounds of the definite integral? 0 and 3

$$\int_0^3 \frac{x}{x^2+1}dx = \int_0^3 \frac{1}{u}\left(\frac{du}{2}\right) = \frac{1}{2}\int_0^3 \frac{1}{u}du$$

$$\frac{1}{2}\ln|u|\Big]_0^3 = \frac{1}{2}\ln|x^2+1|\Big]_0^3$$

$$\frac{1}{2}\left[\ln(3^2+1)-\ln(0^2+1)\right] = \frac{1}{2}(\ln 10 - \ln 1)$$

$$= \frac{1}{2}\ln 10 \approx 1.151$$

$$u = x^2+1$$

$$du = 2xdx$$

$$\frac{du}{2} = xdx$$

Example 4: Recognizing Quotient Forms of the Log Rule

(a). $\int \frac{3x^2+1}{x^3+x}dx = \ln\left|x^3+x\right|+C$

$u = x^3+x$

$du = 3x^2+1dx$

(b). $\int \frac{\sec^2 x}{\tan x}dx = \ln\left|\tan x\right|+C$

$u = \tan x$

$du = \sec^2 xdx$

(c). $\int \frac{x+1}{x^2+2x}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x}dx$

$= \frac{1}{2}\ln\left|x^2+2x\right|+C$

$u = x^2+2x$

$du = 2x+2dx$

(d). $\int \frac{1}{3x+2}dx = \frac{1}{3}\int \frac{3}{3x+2}dx$

$= \frac{1}{3}\ln\left|3x+2\right|+C$

$u = 3x+2$

$du = 3dx$

** Which of the following are equivalent to the antiderivative listed in part (d)?

$\ln\left|(3x+2)^{1/3}\right|+C$ $\quad$ $\frac{1}{3}\ln\left|x+\frac{2}{3}\right|+C$ $\quad$ $\ln\left|3x+2\right|^{1/3}+C$

If a rational function has a numerator that has a degree (exponent) that is greater than or equal to the degree of the denominator, division (yes long division) may reveal a form to which you can apply the Log Rule

Example 5: Using Long Division Before Integrating

$$\int \frac{x^2+x+1}{x^2+1}dx$$

$$\begin{array}{r} 1 \\ 0x+1 \overline{\big) \cancel{x^2}+x+\cancel{1}} \\ -\cancel{x^2} \overset{-}{+} 0x \overset{-}{\cancel{+}}1 \\ \hline x \end{array}$$

Quotient: $1+\frac{x}{x^2+1}$ $\longrightarrow$ Remainder $\longrightarrow$ Divisor

$$= \int\left(1+\frac{x}{x^2+1}\right)dx = \int 1dx + \int\frac{x}{x^2+1}dx$$

$$= x+\frac{1}{2}\int\frac{2x}{x^2+1}dx = x+\frac{1}{2}\ln\left|x^2+1\right|+C$$

$u = x^2+1$

$du = 2xdx$

(15). $\int\frac{x^3-3x^2+5}{x-3}$

** We can use synthetic division here since the divisor has a degree of 1 (it is linear)

$$\begin{array}{r|rrrr} 3 & 1 & -3 & 0 & 5 \\ & & 3 & 0 & 0 \\ \hline & 1 & 0 & 0 & \boxed{5} \end{array}$$

Quotient: $x^2+\frac{5}{x-3}$

$$\int x^2dx+5\int\frac{1}{x-3}dx = \frac{x^3}{3}+5\ln\left|x-3\right|+C$$

(17).

$$\int \frac{x^4+x-4}{x^2+2}\,dx$$

$$= \int \left(x^2-2+\frac{x}{x^2+2} \right) dx$$

$$= \frac{x^3}{3}-2x+\frac{1}{2}\ln\left|x^2+2\right|+C$$

$$u = x^2+2$$

$$du = 2x\,dx$$

$$\begin{array}{r} x^2+0x-2 \\ x^2+0x+2\overline{\big)\,\cancel{x^4}+\cancel{0x^3}+0x^2+x-4} \\ \underline{\cancel{-x^4}\ \cancel{\overset{-}{+}0x^3}\ \overset{-}{+}2x^2} \\ -2x^2+x-4 \\ \underline{+2x^2\ \overset{-}{+}0x+4} \\ x \end{array}$$

Quotient:

$$x^2-2+\frac{x}{x^2+2}$$

Sometimes a change of variables helps to recognize the Log Rule

Example 6: Change of Variables with the Log Rule

$$\int \frac{2\,\text{[illegible]}}{(x+1)^2}\,dx = \int \frac{2(u-1)}{u^2}\,du$$

$$2\int\left(\frac{u}{u^2}-\frac{1}{u^2}\right)du = 2\int\left(\frac{1}{u}-u^{-2}\right)du$$

$$= 2\int\frac{1}{u}\,du - 2\int u^{-2}\,du$$

$$= 2\ln|u| + \frac{2}{u} + C$$

$$= 2\ln|x+1| + \frac{2}{x+1} + C$$

$$u = x+1$$

$$x = u-1$$

$$du = dx$$

** There is an "extra" x variable in this problem - solve u for x

Now, by now you have noticed that integration is not as straight-forward as differentiation - sometimes the integrand is "hidden" and different techniques must be used to "find" it.

Differentiation takes the form: "Here is the question; what is the answer?"
However . . . Integration is more like: "Here is the answer; what is the question?"

Guidelines for Integration

1. Learn a basic list of integration formulas (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trig rules. By the end of Section 5.9, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. If you cannot find a u-substitution that works, try altering the integrand. You might try a trig identity, multiplication and division by the same quantity, or addition or subtraction of the same quantity. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

Example 7: u-Substitution and the Log Rule

Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{x \ln x}$$

$$y = \int \frac{1}{x \ln x} dx = \int \frac{1/x}{\ln x} dx$$

$$= \int \frac{1}{u} du = \ln|u| + C$$

$$= \ln|\ln x| + C$$

There are 3 choices for u:

$u = x$

$u = x \ln x$

Fail to fit u'/u form of the Log Rule

$u = \ln x$

$$du = \frac{1}{x} dx$$

(19). $\int \frac{(\ln x)^2}{x}dx = \int \frac{1}{x}(\ln x)^2 dx$

$u = \ln x$

$= \int u^2 du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C$

$du = \frac{1}{x}dx$

(21). $\int \frac{1}{\sqrt{x+1}}dx = \int (x+1)^{-1/2}dx$

$u = x+1$

$du = dx$

$= \int u^{-1/2}du = \frac{u^{1/2}}{1/2} + C = 2u^{1/2} + C$

$= 2\sqrt{x+1} + C$

Hey!! This problem didn't even use the Log Rule - I guess it wanted to see if you were paying attention!!

(23). $\int \frac{2x}{(x-1)^2}dx = 2\int \frac{x}{(x-1)^2}dx$

$u = x-1$

$du = dx$

$$= 2\int \frac{x+1-1}{(x-1)^2}dx$$

$$= 2\int \frac{(x-1)}{(x-1)^2}dx + 2\int \frac{1}{(x-1)^2}dx$$

$$= 2\int \frac{1}{(x-1)}dx + 2\int \frac{1}{(x-1)^2}dx$$

$$= 2\int \frac{1}{u}dx + 2\int u^{-2}dx$$

$$= 2\ln|u| - \frac{2}{u} + C$$

$$= 2\ln|x-1| - \frac{2}{x-1} + C$$

#25. Find the indefinite integral by u-substitution.
(Hint: Let u be the denominator of the integrand).

$$\int \frac{1}{1+\sqrt{2x}}\,dx = \int \frac{1}{u}(u-1)\,du$$

$$= \int \frac{u-1}{u}\,du = \int\left(\frac{u}{u}-\frac{1}{u}\right)du$$

$$= \int\left(1-\frac{1}{u}\right)du$$

$$= u - \ln|u| + C_1$$

$$= 1+\sqrt{2x} - \ln\left|1+\sqrt{2x}\right| + C_1$$

$$= \sqrt{2x} - \ln\left|1+\sqrt{2x}\right| + C$$

$$C = C_1 + 1$$

Watch this substitution carefully!!!

$$u = 1+\sqrt{2x} = 1+(2x)^{1/2}$$

$$u-1 = \sqrt{2x}$$

$$du = \frac{1}{2}(2x)^{-1/2}(2)$$

$$du = \frac{1}{\sqrt{2x}}\,dx$$

$$\sqrt{2x}\,du = dx$$

$$(u-1)\,du = dx$$

(27). Find the indefinite integral by u-substitution.
Hint: Let u be the denominator of the integrand.

$$\int \frac{\sqrt{x}}{\sqrt{x}-3}dx = \int \frac{(u+3)}{u}\left(2(u+3)du\right)$$

$$= 2\int \frac{(u+3)^2}{u}du = 2\int \frac{u^2+6u+9}{u}du$$

$$= 2\int \frac{u^2}{u}du + 2\int \frac{6u}{u}du + 2\int \frac{9}{u}du$$

$$= 2\int udu + 2\int 6du + 18\int \frac{1}{u}du$$

$$= 2\frac{u^2}{2} + 2(6u) + 18\ln|u| + C_1$$

$$= (\sqrt{x}-3)^2 + 12(\sqrt{x}-3) + 18\ln\left|\sqrt{x}-3\right| + C_1$$

$$= x - 6\sqrt{x} + 9 + 12\sqrt{x} - 36 + 18\ln\left|\sqrt{x}-3\right| + C_1$$

$$= x + 6\sqrt{x} + 18\ln\left|\sqrt{x}-3\right| + C$$

$$C = C_1 + 9 - 36$$

$$u = \sqrt{x} - 3 = x^{\frac{1}{2}} - 3$$

$$u + 3 = \sqrt{x}$$

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx$$

$$du = \frac{1}{2\sqrt{x}}dx$$

$$2\sqrt{x}du = dx$$

$$2(u+3)du = dx$$

Integrals of Trigonometric Functions

Example 8: Using a Trig Identity

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$ Rewrite tan

$u = \cos x$

$du = -\sin x dx$

$-du = \sin x dx$

$$= \int \frac{1}{u}(-du) = -\int \frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

In the next example, you must multiply and divide by the same quantity to derive an integration rule for the secant function.

Example 9: Derivation of the Secant Formula

$$\int \sec x dx = \int \sec x dx \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$ Distribute sec x

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x dx$

$$= \int \frac{1}{u} du = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

Example 10: Integrating Trig Functions

Evaluate $\int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx$ $\qquad 1+\tan^2 x = \sec^2 x$

$$= \int_0^{\pi/4} \sec x\,dx$$

$$= \ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln(\sqrt{2}+1) - \ln 1 \approx 0.881$$

#39. Solve the differential equation. Use the point (0, 2) to find a particular solution

$$\frac{ds}{d\theta} = \tan 2\theta$$

$$s = \int \tan(2\theta)d\theta$$

$$s = \int \tan u\left(\frac{du}{2}\right) = \frac{1}{2}\int \tan u du$$

$$s = -\frac{1}{2}\ln\left|\cos u\right| + C$$

$$s = -\frac{1}{2}\ln\left|\cos 2\theta\right| + C$$

$$(0,2): 2 = -\frac{1}{2}\ln\left|\cos 2(0)\right| + C$$

$$2 = -\frac{1}{2}\ln\left|\cos 0\right| + C$$

$$2 = C$$

$$s = -\frac{1}{2}\ln\left|\cos 2\theta\right| + 2$$

$$u = 2\theta$$

$$du = 2d\theta$$

$$\frac{du}{2} = d\theta$$

Example 11: Finding an Average Value

Find the average value of f(x) on the interval [0, π/4]

$$f(x) = \tan x$$

Average Value $= \dfrac{1}{b-a}\int_a^b f(x)dx$

Average Value $= \dfrac{1}{(\pi/4)-0}\int_0^{\pi/4} \tan x dx$

$$= \frac{4}{\pi}\int_0^{\pi/4} \tan x dx$$

$$= \frac{4}{\pi}[-\ln|\cos x|]_0^{\pi/4}$$

$$= -\frac{4}{\pi}\left[\ln\left|\cos\frac{\pi}{4}\right| - \ln|\cos 0|\right]$$

$$= -\frac{4}{\pi}\left[\ln\left|\frac{\sqrt{2}}{2}\right| - \ln(1)\right]$$

$$= -\frac{4}{\pi}\ln\left(\frac{\sqrt{2}}{2}\right) \approx 0.441$$

HW p330 #2, 6, 12, 16, 18, 20, 26, 28, 34, 40, 42, 44, 78, 80