

# Progressions for the Common Core State Standards in Mathematics (draft)

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# 6–8, Expressions and Equations

## Overview

An *expression* expresses something. Facial expressions express emotions. Mathematical expressions express calculations with numbers. Some of the numbers might be given explicitly, like 2 or  $\frac{3}{4}$ . Other numbers in the expression might be represented by letters, such as  $x$ ,  $y$ ,  $P$ , or  $n$ . The calculation an expression represents might use only a single operation, as in  $4 + 3$  or  $3x$ , or it might use a series of nested or parallel operations, as in  $3(a + 9) - 9/b$ . An expression can consist of just a single number, even 0.

Letters standing for numbers in an expression are called *variables*. In good practice, including in student writing, the meaning of a variable is specified by the surrounding text; an expression by itself gives no intrinsic meaning to the variables in it. Depending on the context, a variable might stand for a specific number, for example the solution to a word problem; it might be used in a universal statement true for all numbers, for example when we say that that  $a + b = b + a$  for all numbers  $a$  and  $b$ ; or it might stand for a range of numbers, for example when we say that  $\sqrt{x^2} = x$  for  $x > 0$ . In choosing variables to represent quantities, students specify a unit; rather than saying “let  $G$  be gasoline,” they say “let  $G$  be the number of gallons of gasoline.”<sup>MP6</sup>

An expression is a phrase in a sentence about a mathematical or real-world situation. As with a facial expression, however, you can read a lot from an algebraic expression (an expression with variables in it) without knowing the story behind it, and it is a goal of this progression for students to see expressions as objects in their own right, and to read the general appearance and fine details of algebraic expressions.

An *equation* is a statement that two expressions are equal, such as  $10 + 0.02n = 20$ , or  $3 + x = 4 + x$ , or  $2(a + 1) = 2a + 2$ . It is an important aspect of equations that the two expressions on either side of the equal sign might not actually always be equal; that is, the equation might be a true statement for some values of the variable(s) and a false statement for others. For example,

MP6 Be precise in defining variables.

$10 + 0.02n = 20$  is true only if  $n = 500$ ; and  $3 + x = 4 + x$  is not true for any number  $x$ ; and  $2(a + 1) = 2a + 2$  is true for all numbers  $a$ . A *solution* to an equation is a number that makes the equation true when substituted for the variable (or, if there is more than one variable, it is a number for each variable). An equation may have no solutions (e.g.,  $3 + x = 4 + x$  has no solutions because, no matter what number  $x$  is, it is not true that adding 3 to  $x$  yields the same answer as adding 4 to  $x$ ). An equation may also have every number for a solution (e.g.,  $2(a + 1) = 2a + 2$ ). An equation that is true no matter what number the variable represents is called an *identity*, and the expressions on each side of the equation are said to be *equivalent expressions*. For example  $2(a + 1)$  and  $2a + 2$  are equivalent expressions. In Grades 6–8, students start to use properties of operations to manipulate algebraic expressions and produce different but equivalent expressions for different purposes. This work builds on their extensive experience in K–5 working with the properties of operations in the context of operations with whole numbers, decimals and fractions.

## Grade 6

**Apply and extend previous understandings of arithmetic to algebraic expressions** Students have been writing numerical expressions since Kindergarten, such as

$$2 + 3 \quad 7 + 6 + 3 \quad 4 \times (2 \times 3)$$

$$8 \times 5 + 8 \times 2 \quad \frac{1}{3}(8 + 7 + 3) \quad \frac{3}{\frac{1}{2}}$$

In Grade 5 they used whole number exponents to express powers of 10, and in Grade 6 they start to incorporate whole number exponents into numerical expressions, for example when they describe a square with side length 50 feet as having an area of  $50^2$  square feet.<sup>6.EE.1</sup>

Students have also been using letters to represent an unknown quantity in word problems since Grade 3. In Grade 6 they begin to work systematically with algebraic expressions. They express the calculation “Subtract  $y$  from 5” as  $5 - y$ , and write expressions for repeated numerical calculations.<sup>MP8</sup> For example, students might be asked to write a numerical expression for the change from a \$10 bill after buying a book at various prices:

|                    |          |             |             |
|--------------------|----------|-------------|-------------|
| Price of book (\$) | 5.00     | 6.49        | 7.15        |
| Change from \$10   | $10 - 5$ | $10 - 6.49$ | $10 - 7.15$ |

Abstracting the pattern they write  $10 - p$  for a book costing  $p$  dollars, thus summarizing a calculation that can be carried out repeatedly with different numbers.<sup>6.EE.2a</sup> Such work also helps students interpret expressions. For example, if there are 3 loose apples and 2 bags of  $A$  apples each, students relate quantities in the situation to the terms in the expression  $3 + 2A$ .

As they start to solve word problems algebraically, students also use more complex expressions. For example, in solving the word problem

Daniel went to visit his grandmother, who gave him \$5.50. Then he bought a book costing \$9.20. If he has \$2.30 left, how much money did he have before visiting his grandmother?

students might obtain the expression  $x + 5.50 - 9.20$  by following the story forward, and then solve the equation  $x + 5.50 - 9.20 = 2.30$ .<sup>•</sup> Students may need explicit guidance in order to develop the strategy of working forwards, rather than working backwards from the 2.30 and calculating  $2.30 + 9.20 - 5.50$ .<sup>6.EE.7</sup> As word problems get more complex, students find greater benefit in representing the problem algebraically by choosing variables to represent quantities, rather than attempting a direct numerical solution, since the former approach provides general methods and relieves demands on working memory.

**6.EE.1** Write and evaluate numerical expressions involving whole-number exponents.

**MP8** Look for regularity in a repeated calculation and express it with a general formula.

**6.EE.2a** Write, read, and evaluate expressions in which letters stand for numbers.

a Write expressions that record operations with numbers and with letters standing for numbers.

• Notice that in this problem, like many problems, a quantity, “money left,” is expressed in two distinct ways:

1. starting amount + amount from grandma – amount spent
2. \$2.30

Because these two expressions refer to the same quantity in the problem situation, they are equal to each other. The equation formed by representing their equality can then be solved to find the unknown value (that is, the value of the variable that makes the equation fit the situation).

**6.EE.7** Solve real-world and mathematical problems by writing and solving equations of the form  $x + p = q$  and  $px = q$  for cases in which  $p$ ,  $q$  and  $x$  are all nonnegative rational numbers.

Students in Grade 5 began to move from viewing expressions as actions describing a calculation to viewing them as objects in their own right;<sup>5.OA.2</sup> in Grade 6 this work continues and becomes more sophisticated. They describe the structure of an expression, seeing  $2(8 + 7)$  for example as a product of two factors the second of which,  $(8 + 7)$ , can be viewed as both a single entity and a sum of two terms. They interpret the structure of an expression in terms of a context: if a runner is  $7t$  miles from her starting point after  $t$  hours, what is the meaning of the  $7$ ?<sup>MP7</sup> If  $a$ ,  $b$ , and  $c$  are the heights of three students in inches, they recognize that the coefficient  $\frac{1}{3}$  in  $\frac{1}{3}(a + b + c)$  has the effect of reducing the size of the sum, and they also interpret multiplying by  $\frac{1}{3}$  as dividing by 3.<sup>6.EE.2b</sup> Both interpretations are useful in connection with understanding the expression as the mean of  $a$ ,  $b$ , and  $c$ .<sup>6.SP.3</sup>

In the work on number and operations in Grades K–5, students have been using properties of operations to write expressions in different ways. For example, students in grades K–5 write  $2 + 3 = 3 + 2$  and  $8 \times 5 + 8 \times 2 = 8 \times (5 + 2)$  and recognize these as instances of general properties which they can describe. They use the “any order, any grouping” property<sup>•</sup> to see the expression  $7 + 6 + 3$  as  $(7 + 3) + 6$ , allowing them to quickly evaluate it. The properties are powerful tools that students use to accomplish what they want when working with expressions and equations. They can be used at any time, in any order, whenever they serve a purpose.

Work with numerical expressions prepares students for work with algebraic expressions. During the transition, it can be helpful for them to solve numerical problems in which it is more efficient to hold numerical expressions unevaluated at intermediate steps. For example, the problem

Fred and George Weasley make 150 “Deflagration Deluxe” boxes of Weasleys’ Wildfire Whiz-bangs at a cost of 17 Galleons each, and sell them for 20 Galleons each. What is their profit?

is more easily solved by leaving unevaluated the total cost,  $150 \times 17$  Galleons, and the total revenue  $150 \times 20$  Galleons, until the subtraction step, where the distributive law can be used to calculate the answer as  $150 \times 20 - 150 \times 17 = 150 \times 3 = 450$  Galleons. A later algebraic version of the problem might ask for the sale price that will yield a given profit, with the sale price represented by a letter such as  $p$ . The habit of leaving numerical expressions unevaluated prepares students for constructing the appropriate algebraic equation to solve such a problem.

As students move from numerical to algebraic work the multiplication and division symbols  $\times$  and  $\div$  are replaced by the conventions of algebraic notation. Students learn to use either a dot for multiplication, e.g.,  $1 \cdot 2 \cdot 3$  instead of  $1 \times 2 \times 3$ , or simple juxtaposition, e.g.,  $3x$  instead of  $3 \times x$  (during the transition, students may indicate all multiplications with a dot, writing  $3 \cdot x$  for  $3x$ ). A firm grasp

*Draft, 4/22/2011, comment at [commoncoretools.wordpress.com](http://commoncoretools.wordpress.com).*

5.OA.2 Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

MP7 Looking for structure in expressions by parsing them into a sequence of operations; making use of the structure to interpret the expression’s meaning in terms of the quantities represented by the variables.

6.EE.2b Write, read, and evaluate expressions in which letters stand for numbers.

b Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity.

6.SP.3 Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.

• The “any order, any grouping” property is a combination of the commutative and associative properties. It says that sequence of additions and subtractions may be calculated in any order, and that terms may be grouped together any way.

#### Some common student difficulties

- Failure to see juxtaposition as indicating multiplication, e.g., evaluating  $3x$  as 35 when  $x = 5$ , or rewriting  $8 - 2a$  as  $6a$
- Failure to see hidden 1s, rewriting  $4C - C$  as 4 instead of seeing  $4C - C$  as  $4 \cdot C - 1 \cdot C$  which is  $3 \cdot C$ .

on variables as numbers helps students extend their work with the properties of operations from arithmetic to algebra.<sup>MP2</sup> For example, students who are accustomed to mentally calculating  $5 \times 37$  as  $5 \times (30 + 7) = 150 + 35$  can now see that  $5(3a + 7) = 15a + 35$  for all numbers  $a$ . They apply the distributive property to the expression  $3(2 + x)$  to produce the equivalent expression  $6 + 3x$  and to the expression  $24x + 18y$  to produce the equivalent expression  $6(4x + 3y)$ .<sup>6.EE.3</sup>

Students evaluate expressions that arise from formulas used in real-world problems, such as the formulas  $V = s^3$  and  $A = 6s^2$  for the volume and surface area of a cube. In addition to using the properties of operations, students use conventions about the order in which arithmetic operations are performed in the absence of parentheses.<sup>6.EE.2c</sup> It is important to distinguish between such conventions, which are notational conveniences that allow for algebraic expressions to be written with fewer parentheses, and properties of operations, which are fundamental properties of the number system and undergird all work with expressions. In particular, the mnemonic PEMDAS<sup>•</sup> can mislead students into thinking, for example, that addition must always take precedence over subtraction because the A comes before the S, rather than the correct convention that addition and subtraction proceed from left to right (as do multiplication and division). This can lead students to make mistakes such as simplifying  $n - 2 + 5$  as  $n - 7$  (instead of the correct  $n + 3$ ) because they add the 2 and the 5 before subtracting from  $n$ .<sup>6.EE.4</sup>

The order of operations tells us how to interpret expressions, but does not necessarily dictate how to calculate them. For example, the P in PEMDAS indicates that the expression  $8 \times (5 + 1)$  is to be interpreted as 8 times a number which is the sum of 5 and 1. However, it does not dictate the expression must be calculated this way. A student might well see it, through an implicit use of the distributive law, as  $8 \times 5 + 8 \times 1 = 40 + 8 = 48$ .

The distributive law is of fundamental importance. Collecting like terms, e.g.,  $5b + 3b = (5 + 3)b = 8b$ , should be seen as an application of the distributive law, not as a separate method.

### Reason about and solve one-variable equations and inequalities

In Grades K–5 students have been writing numerical equations and simple equations involving one operation with a variable. In Grade 6 they start the systematic study of equations and inequalities and methods of solving them. Solving is a process of reasoning to find the numbers which make an equation true, which can include checking if a given number is a solution.<sup>6.EE.5</sup> Although the process of reasoning will eventually lead to standard methods for solving equations, students should study examples where looking for structure pays off, such as in  $4x + 3x = 3x + 20$ , where they can see that  $4x$  must be 20 to make the two sides equal.

This understanding can be reinforced by comparing arithmetic

MP2 Connect abstract symbols to their numerical referents.

6.EE.3 Apply the properties of operations to generate equivalent expressions.

6.EE.2c Write, read, and evaluate expressions in which letters stand for numbers.

c Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

• PEMDAS stands for Parentheses, Exponents, Multiplication, Division, Addition, Subtraction, specifying the order in which operations are performed in interpreting or evaluating numerical expressions.

6.EE.4 Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them).

6.EE.5 Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

and algebraic solutions to simple word problems. For example, how many 44-cent stamps can you buy with \$11? Students are accustomed to solving such problems by division; now they see the parallel with representing the problem algebraically as  $0.44n = 11$ , from which they use the same reasoning as in the numerical solution to conclude that  $n = 11 \div 0.44$ .<sup>6.EE.7</sup> They explore methods such as dividing both sides by the same non-zero number. As word problems grow more complex in Grades 6 and 7, analogous arithmetical and algebraic solutions show the connection between the procedures of solving equations and the reasoning behind those procedures.

When students start studying equations in one variable, it is important for them to understand every occurrence of a given variable has the same value in the expression and throughout a solution procedure: if  $x$  is assumed to be the number satisfying the equation  $4x + 3x = 3x + 20$  at the beginning of a solution procedure, it remains that number throughout.

As with all their work with variables, it is important for students to state precisely the meaning of variables they use when setting up equations (MP6). This includes specifying whether the variable refers to a specific number, or to all numbers in some range. For example, in the equation  $0.44n = 11$  the variable  $n$  refers to a specific number (the number of stamps you can buy for \$11); however, if the expression  $0.44n$  is presented as a general formula for calculating the price in dollars of  $n$  stamps, then  $n$  refers to all numbers in some domain.<sup>6.EE.6</sup> That domain might be specified by inequalities, such as  $n > 0$ .<sup>6.EE.8</sup>

**Represent and analyze quantitative relationships between dependent and independent variables** In addition to constructing and solving equations in one variable, students use equations in two variables to express relationships between two quantities that vary together. When they construct an expression like  $10 - p$  to represent a quantity such as on page 4, students can choose a variable such as  $C$  to represent the calculated quantity and write  $C = 10 - p$  to represent the relationship. This prepares students for work with functions in later grades.<sup>6.EE.9</sup> The variable  $p$  is the natural choice for the independent variable in this situation, with  $C$  the dependent variable. In a situation where the price,  $p$ , is to be calculated from the change,  $C$ , it might be the other way around.

As they work with such equations students begin to develop a dynamic understanding of variables, an appreciation that they can stand for any number from some domain. This use of variables arises when students study expressions such as  $0.44n$ , discussed earlier, or equations in two variables such as  $d = 5 + 5t$  describing relationship between distance in miles,  $d$ , and time in hours,  $t$ , for a person starting 5 miles from home and walking away at 5 miles per hour. Students can use tabular<sup>•</sup> and graphical<sup>•</sup> representations to develop an appreciation of varying quantities.

**6.EE.7** Solve real-world and mathematical problems by writing and solving equations of the form  $x + p = q$  and  $px = q$  for cases in which  $p$ ,  $q$  and  $x$  are all nonnegative rational numbers.

• In Grade 7, where students learn about complex fractions, this problem can be expressed in cents as well as dollars to help students understand equivalences such as

$$\frac{11}{0.44} = \frac{1100}{44}$$

#### Analogous arithmetical and algebraic solutions

J. bought three packs of balloons. He opened them and counted 12 balloons. How many balloons are in a pack?

*Arithmetical solution*

If three packs have twelve balloons, then one pack has  $12 \div 3 = 4$  balloons.

*Algebraic solution*

Defining the variable: Let  $b$  be the number of balloons in a pack. Writing the equation:

$$3b = 12$$

Solving (mirrors the reasoning of the numerical solution):

$$3b = 12 \rightarrow \frac{3b}{3} = \frac{12}{3}$$

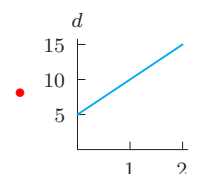
$$b = 4.$$

**6.EE.6** Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

**6.EE.8** Write an inequality of the form  $x > c$  or  $x < c$  to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form  $x > c$  or  $x < c$  have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

**6.EE.9** Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

|         |      |      |      |      |      |
|---------|------|------|------|------|------|
| $n$     | 1    | 2    | 3    | 4    | 5    |
| $0.44n$ | 0.44 | 0.88 | 1.32 | 1.75 | 2.20 |



## Grade 7

### Use properties of operations to generate equivalent expressions

In Grade 7 students start to simplify general linear expressions with rational coefficients. Building on work in Grade 6, where students used conventions about the order of operations to parse, and properties of operations to transform, simple expressions such as  $2(3 + 8x)$  or  $10 - 2p$ , students now encounter linear expressions with more operations and whose transformation may require an understanding of the rules for multiplying negative numbers, such as  $7 - 2(3 - 8x)$ .<sup>7.EE.1</sup>

In simplifying this expression students might come up with answers such as

- $5(3 - 8x)$ , mistakenly detaching the 2 from the indicated multiplication
- $7 - 2(-5x)$ , through a determination to perform the computation in parentheses first, even though no simplification is possible
- $7 - 6 - 16x$ , through an imperfect understanding of the way the distributive law works or of the rules for multiplying negative numbers.

In contrast with the simple linear expressions they see in Grade 6, the more complex expressions students seen in Grade 7 afford shifts of perspective, particularly because of their experience with negative numbers: for example, students might see  $7 - 2(3 - 8x)$  as  $7 - (2(3 - 8x))$  or as  $7 + (-2)(3 + (-8)x)$  (MP7).

As students gain experience with multiple ways of writing an expression, they also learn that different ways of writing expressions can serve different purposes and provide different ways of seeing a problem. For example,  $a + 0.05a = 1.05a$  means that “increase by 5%” is the same as “multiply by 1.05.”<sup>7.EE.2</sup> In the example on the right, the connection between the expressions and the figure emphasize that they all represent the same number, and the connection between the structure of each expression and a method of calculation emphasize the fact that expressions are built up from operations on numbers.

**Solve real-life and mathematical problems using numerical and algebraic expressions and equations** By Grade 7 students start to see whole numbers, integers, and positive and negative fractions as belonging to a single system of rational numbers, and they solve multi-step problems involving rational numbers presented in various forms.<sup>7.EE.3</sup>

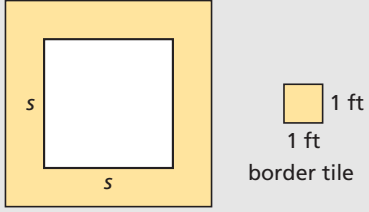
Students use mental computation and estimation to assess the reasonableness of their solutions. For example, the following statement appeared in an article about the annual migration of the Bar-tailed Godwit from Alaska to New Zealand:

*Draft, 4/22/2011, comment at [commoncoretools.wordpress.com](http://commoncoretools.wordpress.com).*

- A *general linear expression* in the variable  $x$  is a sum of terms which are either rational numbers, or rational numbers times  $x$ , e.g.,  $-\frac{1}{2} + 2x + \frac{5}{8} + 3x$ .

**7.EE.1** Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

**Writing expressions in different forms**



*In expressing the number of tiles needed to border a square pool with side length  $s$  feet (where  $s$  is a whole number), students might write  $4(s + 1)$ ,  $s + s + s + s + 4$ , or  $2s + 2(s + 2)$ , each indicating a different way of breaking up the border in order to perform the calculation. They should see all these expressions as equivalent.*

**7.EE.2** Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related.

**7.EE.3** Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. . . .



She had flown for eight days—nonstop—covering approximately 7,250 miles at an average speed of nearly 35 miles per hour.

Students can make the rough mental estimate

$$8 \times 24 \times 35 = 8 \times 12 \times 70 < 100 \times 70 = 7000$$

to recognize that although this astonishing statement is in the right ballpark, the average speed is in fact greater than 35 miles per hour, suggesting that one of the numbers in the article must be wrong.<sup>7.EE.3</sup>

As they build a systematic approach to solving equations in one variable, students continue to compare arithmetical and algebraic solutions to word problems. For example they solve the problem

The perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

by subtracting  $2 \cdot 6$  from 54 and dividing by 2, and also by setting up the equation

$$2w + 2 \cdot 6 = 54.$$

The steps in solving the equation mirror the steps in the numerical solution. As problems get more complex, algebraic methods become more valuable. For example, in the cyclist problem in the margin, the numerical solution requires some insight in order to keep the cognitive load of the calculations in check. By contrast, choosing the letter  $s$  to stand for the unknown speed, students build an equation by adding the distances travelled in three hours ( $3s$  and  $3 \cdot 12.5$ ) and setting them equal to 63 to get

$$3s + 3 \cdot 12.5 = 63.$$

It is worthwhile exploring two different possible next steps in the solution of this equation:

$$3s + 37.5 = 64 \quad \text{and} \quad 3(s + 12.5) = 63.$$

The first is suggested by a standard approach to solving linear equations; the second is suggested by a comparison with the numerical solution described earlier.<sup>7.EE.4a</sup>

Students also set up and solve inequalities, recognizing the ways in which the process of solving them is similar to the process of solving linear equations:

As a salesperson, you are paid \$50 per week plus \$3 per sale. This week you want your pay to be at least \$100. Write an inequality for the number of sales you need to make, and describe the solution.

7.EE.3 ... Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

#### Looking for structure in word problems (MP7)

Two cyclists are riding toward each other along a road (each at a constant speed). At 8 am, they are 63 miles apart. They meet at 11 am. If one cyclist rides at 12.5 miles per hour, what is the speed of the other cyclist?

*First solution: The first cyclist travels  $3 \times 12.5 = 37.5$  miles. The second travels  $63 - 37.5 = 25.5$  miles, so goes  $\frac{25.5}{3} = 8.5$  miles per hour. Another solution uses a key hidden quantity, the speed at which the cyclists are approaching each other, to simplify the calculations: since  $\frac{63}{3} = 21$ , the cyclists are approaching each other at 21 miles per hour, so the other cyclist is traveling at  $21 - 12.5 = 8.5$  miles per hour.*

7.EE.4a Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

- a Solve word problems leading to equations of the form  $px + q = r$  and  $p(x + q) = r$ , where  $p$ ,  $q$ , and  $r$  are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach.

Students also recognize one important new consideration in solving inequalities: multiplying or dividing both sides of an inequality by a negative number reverses the order of the comparison it represents. It is useful to present contexts that allows students to make sense of this. For example,

If the price of a ticket to a school concert is  $p$  dollars then the attendance is  $1000 - 50p$ . What range of prices ensures that at least 600 people attend?

Students recognize that the requirement of at least 600 people leads to the inequality  $1000 - 50p \geq 600$ . Before solving the inequality, they use common sense to anticipate that that answer will be of the form  $p \leq ?$ , since higher prices result in lower attendance.<sup>7.EE.4b</sup> (Note that inequalities using  $\leq$  and  $\geq$  are included in this standard, in addition to  $>$  and  $<$ .)

**7.EE.4b** Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

- b Solve word problems leading to inequalities of the form  $px + q > r$  or  $px + q < r$ , where  $p$ ,  $q$ , and  $r$  are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem.

## Grade 8

**Work with radicals and integer exponents** In Grade 8 students add the properties of integer exponents to their repertoire of rules for transforming expressions. Students have been denoting whole number powers of 10 with exponential notation since Grade 5, and they have seen the pattern in the number of zeros when powers of 10 are multiplied. They express this as  $10^a 10^b = 10^{a+b}$  for whole numbers  $a$  and  $b$ . Requiring this rule to hold when  $a$  and  $b$  are integers leads to the definition of the meaning of powers with 0 and negative exponents. For example, we define  $10^0 = 1$  because we want  $10^a 10^0 = 10^{a+0} = 10^a$ , so  $10^0$  must equal 1. Students extend these rules to other bases, and learn other properties of exponents.<sup>8.EE.1</sup>

Notice that students do not learn the properties of rational exponents until high school. However, they prepare in Grade 8 by starting to work systematically with the square root and cube root symbols, writing, for example,  $\sqrt{64} = \sqrt{8^2} = 8$  and  $(\sqrt[3]{5})^3 = 5$ . Since  $\sqrt{p}$  is defined to mean the positive solution to the equation  $x^2 = p$  (when it exists), it is not correct to say (as is common) that  $\sqrt{64} = \pm 8$ . On the other hand, in describing the solutions to  $x^2 = 64$ , students can write  $x = \pm \sqrt{64} = \pm 8$ .<sup>8.EE.2</sup> Students in Grade 8 are not in a position to prove that these are the only solutions, but rather use informal methods such as guess and check.

Students gain experience with the properties of exponents by working with estimates of very large and very small quantities. For example, they estimate the population of the United States as  $3 \times 10^8$  and the population of the world as  $7 \times 10^9$ , and determine that the world population is more than 20 times larger.<sup>8.EE.3</sup> They express and perform calculations with very large numbers using scientific notation. For example, given that we breathe about 6 liters of air per minute, they estimate that there are  $60 \times 24 = 6 \times 2.4 \times 10^2 \approx 1.5 \times 10^3$  minutes in a day, and that we therefore breath about  $6 \times 1.5 \times 10^3 \approx 10^4$  liters in a day. In a lifetime of 75 years there are about  $365 \times 75 \approx 3 \times 10^4$  days, and so we breath about  $3 \times 10^4 \times 10^4 = 3 \times 10^8$  liters of air in a lifetime.<sup>8.EE.4</sup>

**Understand the connections between proportional relationships, lines, and linear equations** As students in Grade 8 move towards an understanding of the idea of a function, they begin to tie together a number of notions that have been developing over the last few grades:

1. An expression in one variable defines a general calculation in which the variable can represent a range of numbers—an input-output machine with the variable representing the input and the expression calculating the output. For example,  $60t$  is the distance traveled in  $t$  hours by a car traveling at a constant speed of 60 miles per hour.

• **Properties of Integer Exponents**  
For any nonzero rational numbers  $a$  and  $b$  and integers  $n$  and  $m$ :

1.  $a^n a^m = a^{n+m}$
2.  $(a^n)^m = a^{nm}$
3.  $a^n b^n = (ab)^n$
4.  $a^0 = 1$
5.  $a^{-n} = 1/a^n$

**8.EE.1** Know and apply the properties of integer exponents to generate equivalent numerical expressions.

**8.EE.2** Use square root and cube root symbols to represent solutions to equations of the form  $x^2 = p$  and  $x^3 = p$ , where  $p$  is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that  $\sqrt{2}$  is irrational.

**8.EE.3** Use numbers expressed in the form of a single digit times a whole-number power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other.

**8.EE.4** Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

2. Choosing a variable to represent the output leads to an equation in two variables describing the relation between two quantities. For example, choosing  $d$  to represent the distance traveled by the car traveling at 65 miles per hour yields the equation  $d = 65t$ . Reading the expression on the right (multiplication of the variable by a constant) reveals the relationship (a rate relationship in which distance is proportional to time).
3. Tabulating values of the expression is the same as tabulating solution pairs of the corresponding equation. This gives insight into the nature of the relationship; for example, that the distance increases by the same amount for the same increase in the time (the ratio between the two being the speed).
4. Plotting points on the coordinate plane, in which each axis is marked with a scale representing one quantity, affords a visual representation of the relationship between two quantities.

Proportional relationships provide a fruitful first ground in which these notions can grow together. The constant of proportionality is visible in each; as the multiplicative factor in the expression, as the slope of the line, and as an increment in the table (if the dependent variable goes up by 1 unit in each entry). As students start to build a unified notion of the concept of function they are able to compare proportional relationships presented in different ways. For example, the table shows 300 miles in 5 hours, whereas the graph shows more than 300 miles in the same time.<sup>8.EE.5</sup>

The connection between the unit rate in a proportional relationship and the slope of its graph depends on a connection with the geometry of similar triangles. The fact that a line has a well-defined slope—that the ratio between the rise and run for any two points on the line is always the same—depends on similar triangles.<sup>8.EE.6</sup>

The fact that the slope is constant between any two points on a line leads to the derivation of an equation for the line. For a line through the origin, the right triangle whose hypotenuse is the line segment from  $(0,0)$  to a point  $(x,y)$  on the line is similar to the right triangle from  $(0,0)$  to the point  $(1,m)$  on the line, and so

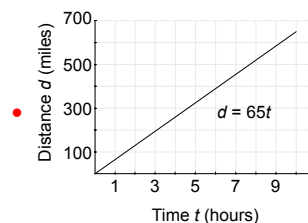
$$\frac{y}{x} = \frac{m}{1}, \quad \text{or} \quad y = mx.$$

The equation for a line not through the origin can be derived in a similar way, starting from the  $y$ -intercept  $(0, b)$  instead of the origin.

**Analyze and solve linear equations and pairs of simultaneous linear equations** By Grade 8 students have the tools to solve an equation which has a general linear expression on each side of the equal sign,<sup>8.EE.7</sup> for example:

If a bar of soap balances  $\frac{3}{4}$  of a bar of soap and  $\frac{3}{4}$  of a pound, how much does the bar of soap weigh?

|               |    |     |     |     |     |     |
|---------------|----|-----|-----|-----|-----|-----|
| $t$ (hours)   | 1  | 2   | 3   | 4   | 5   | 6   |
| $60t$ (miles) | 60 | 120 | 180 | 240 | 300 | 360 |



- In the Grade 8 Functions domain, students see the relationship between the graph of a proportional relationship and its equation  $y = mx$  as a special case of the relationship between a line and its equation  $y = mx + b$ , with  $b = 0$ .

**8.EE.5** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

**Why lines have constant slope**

The green triangle is similar to the blue triangle because corresponding angles are equal, so the ratio of rise to run is the same in each.

**8.EE.6** Use similar triangles to explain why the slope  $m$  is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation  $y = mx$  for a line through the origin and the equation  $y = mx + b$  for a line intercepting the vertical axis at  $b$ .

**8.EE.7** Solve linear equations in one variable.

This is an example where choosing a letter, say  $b$ , to represent the weight of the bar of soap and solving the equation

$$b = \frac{3}{4}b + \frac{3}{4}$$

is probably easier for students than reasoning through a numerical solution. Linear equations also arise in problems where two linear functions are compared. For example

Henry and Jose are gaining weight for football. Henry weighs 205 pounds and is gaining 2 pounds per week. Jose weighs 195 pounds and is gaining 3 pounds per week. When will they weigh the same?

Students in Grade 8 also start to solve problems that lead to simultaneous equations,<sup>8.EE.8</sup> for example

Tickets for the class show are \$3 for students and \$10 for adults. The auditorium holds 450 people. The show was sold out and the class raised \$2750 in ticket sales. How many students bought tickets?

This problem involves two variables, the number  $S$  of student tickets sold and the number  $A$  of adult tickets sold, and imposes two constraints on those variables: the number of tickets sold is 450 and the dollar value of tickets sold is 2750.

8.EE.8 Analyze and solve pairs of simultaneous linear equations.